Algorithms for Causal Probabilistic Graphical Models

Class 3:

Search

Athens Summer School on AI

July 2024

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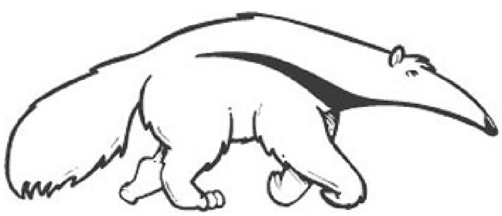

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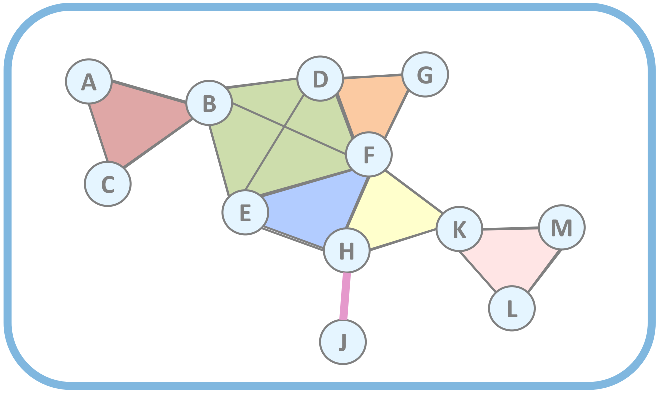
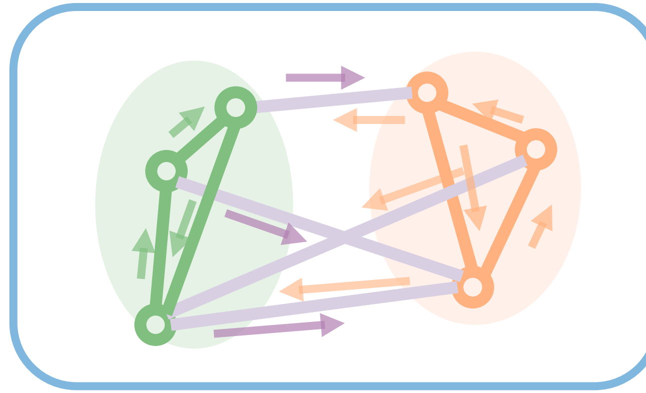
Outline of Lectures

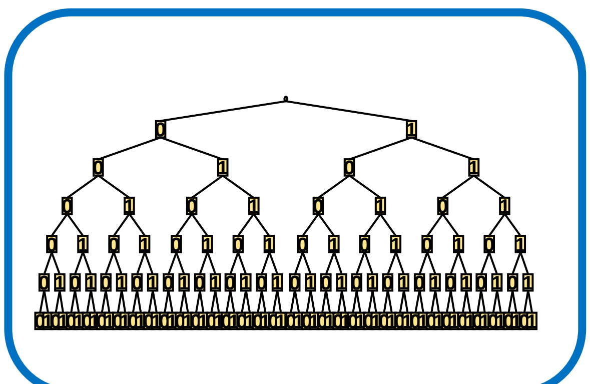
Class 1: Introduction & Inference



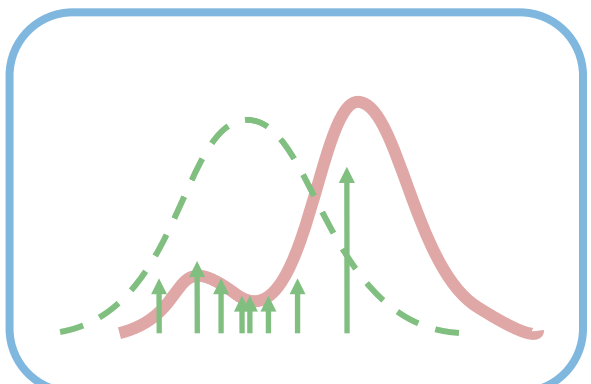
Class 2: Bounds & Variational Methods



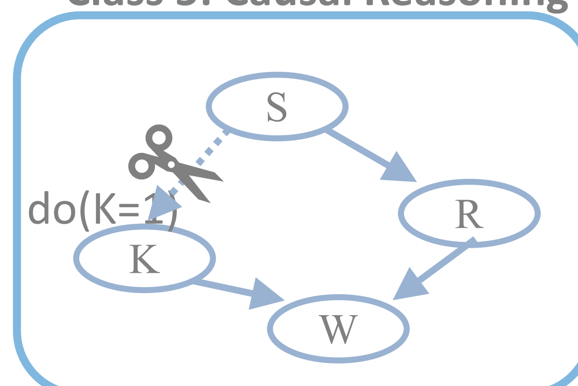
Class 3: Search Methods



Class 4: Monte Carlo Methods



Class 5: Causal Reasoning



Graphical Models

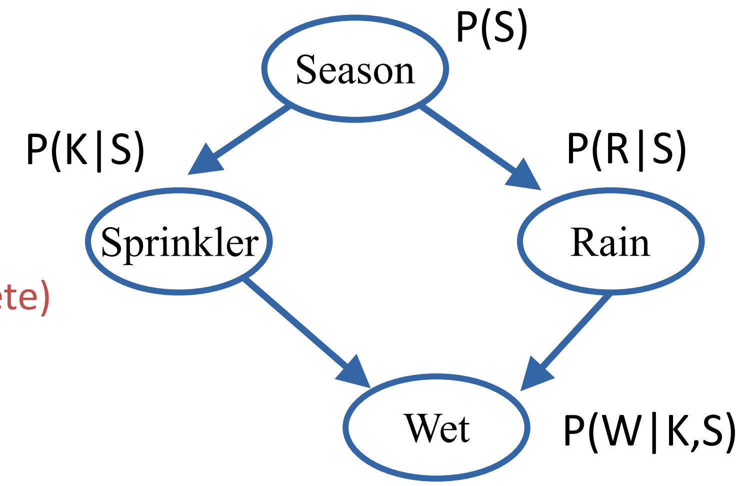
A *graphical model* consists of:

$X = \{X_1, \ldots, X_n\}$ -- variables

$D = \{D_1, \ldots, D_n\}$ -- domains (we'll assume discrete)

$F = \{f_{\alpha_1}, \ldots, f_{\alpha_m}\}$ -- functions or CPTs

and a *combination operator*



The *combination operator* defines an overall function from the individual factors,

e.g., "+" : $P(S, K, R, W) = P(S) \cdot P(K|S) \cdot P(R|S) \cdot P(W|K, S)$

Notation:

Discrete X_i values called "states"

"Tuple" or "configuration": states taken by a set of variables

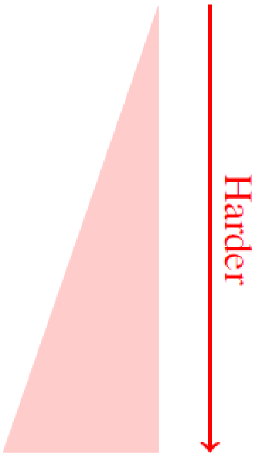
"Scope" of f: set of variables that are arguments to a factor f

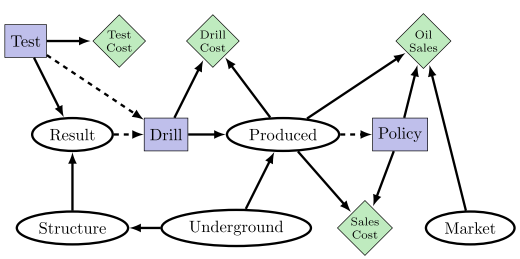
often index factors by their scope, e.g., $f_\alpha(X_\alpha), \quad X_\alpha \subseteq X$

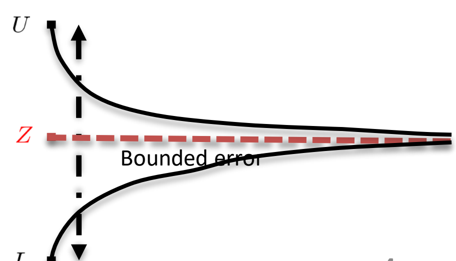
Probabilistic Reasoning Problems

- Exact inference time, space exponential in induced width
- Use search to trade memory for time and time for anytime bounds.

Max-Inference:	$f(x^*) = \max_x \prod_\alpha f_\alpha(x_\alpha)$
Sum-Inference: (e.g., causal effects)	$Z = \sum_x \prod_\alpha f_\alpha(x_\alpha)$
Mixed-Inference (MMAP):	$f_M(x_M^*) = \max_{x_M} \sum_{x_S} \prod_\alpha f_\alpha(x_\alpha)$
Mixed-Inference (MEU): (e.g., decisions, planning)	$\text{MEU} = \max_{D_1,\ldots,D_m} \sum_{X_1,\ldots X_n} (\prod_{P_i \in P} P_i) \times (\sum_{r_i \in R} r_i)$







Outline: Search

- AND/OR Search Trees
- AND/OR Search Graphs
- Pseudo trees generation

AND/OR Search spaces

- Basic Search (depth and Best)
- AND/OR Depth and Best Heuristic Search
- The Guiding MBE Heuristic
- Searching for Mixed tasks

AND/OR Heuristic Search

- Hybrid of Search and Inference

Search & Inference

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AND/OR Heuristic Search

- Hybrid of Search and Inference

Search & Inference

The Probability Tree

$$P(a,e=0) = P(a)\sum_b P(b|a)\sum_c P(c|a)\sum_b P(d|a,b)\sum_{e=0} P(e|b,c)$$

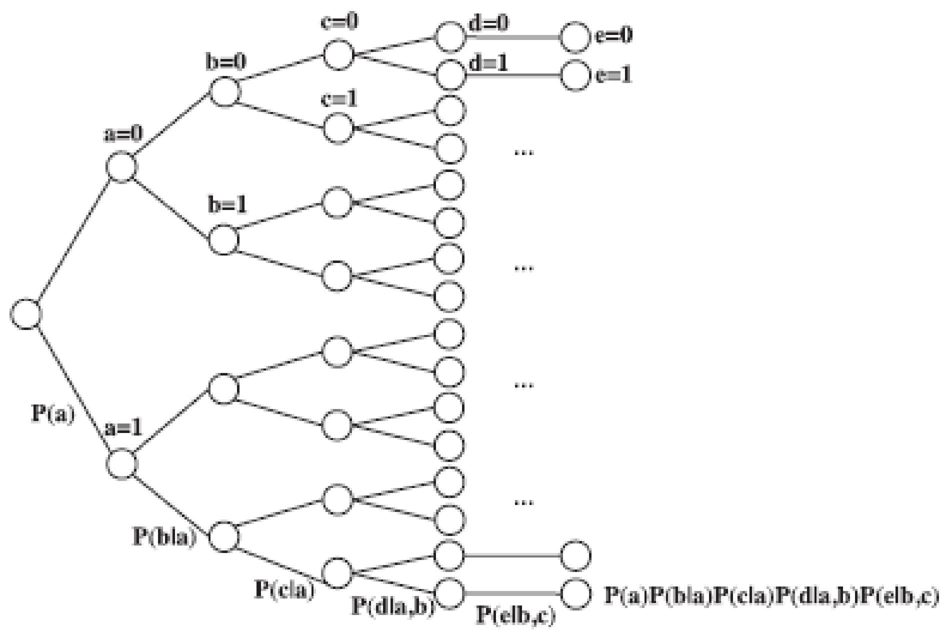
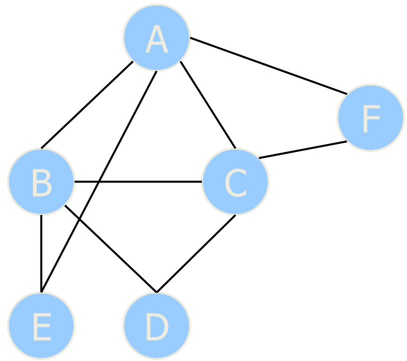


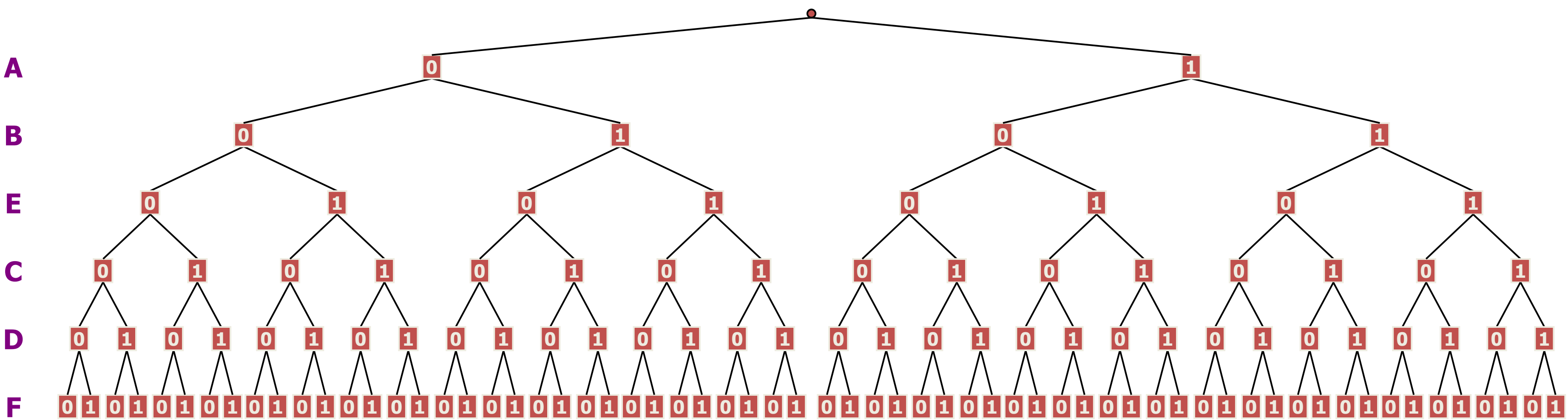
Figure 6.1: Probability tree for computing P(d=1,g=0).

Complexity of conditioning: exponential time, linear space

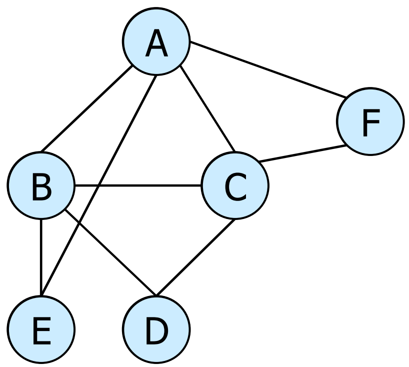
The Classic OR Search Space



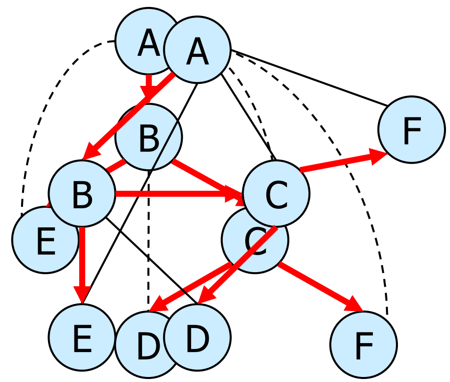
Ordering: A B E C D F



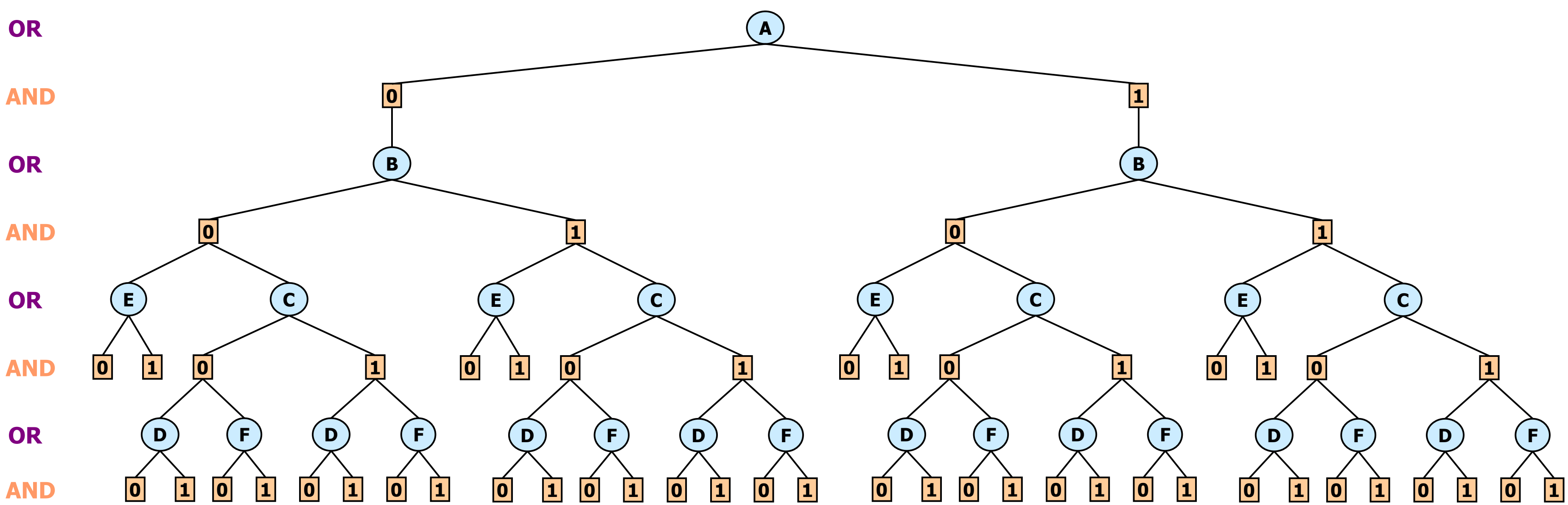
AND/OR Search Space



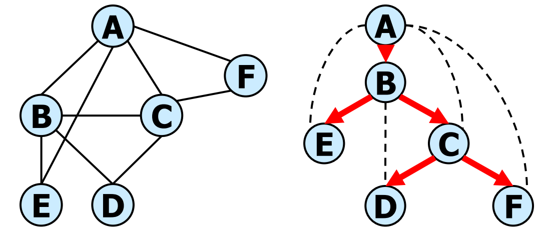
Primal graph

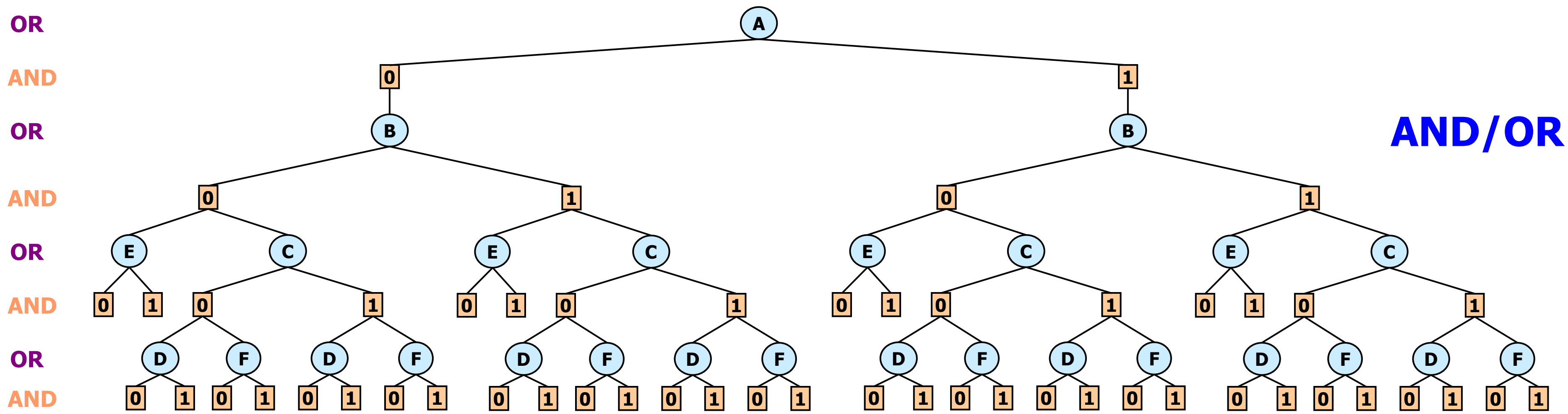


DFS tree

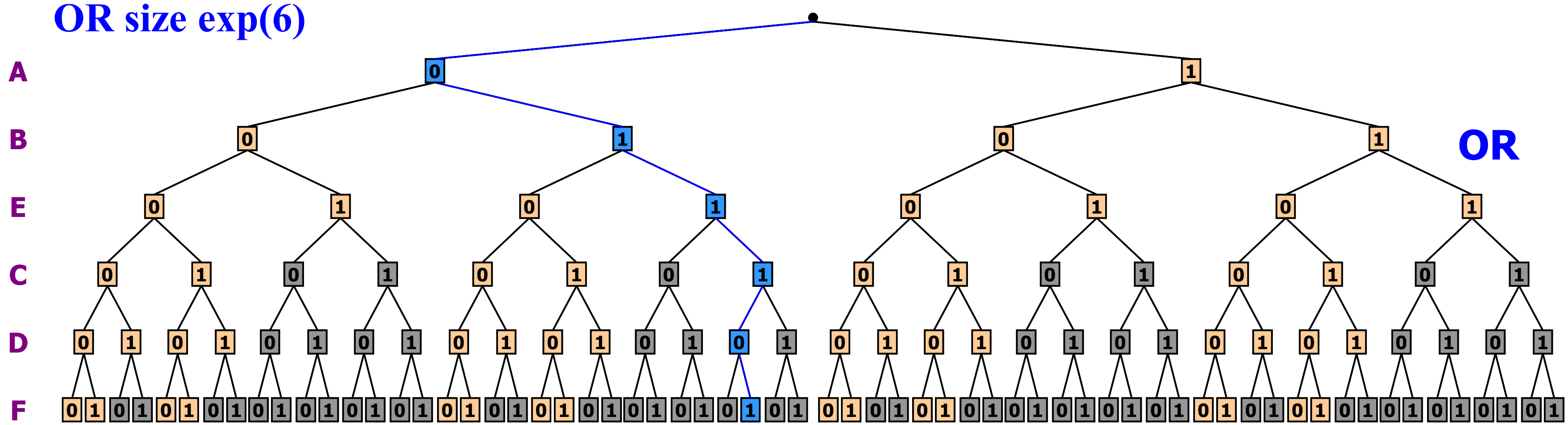


AND/OR vs. OR

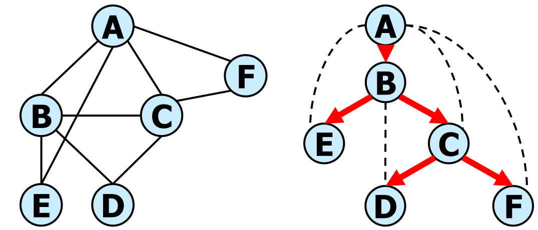


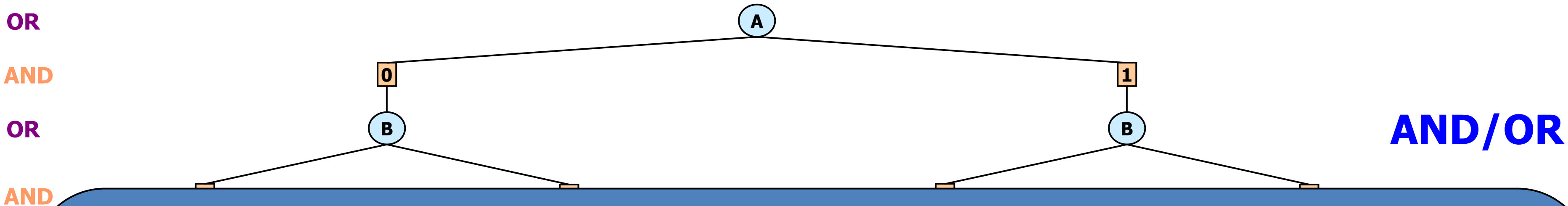


AND/OR size: exp(4),
OR size exp(6)

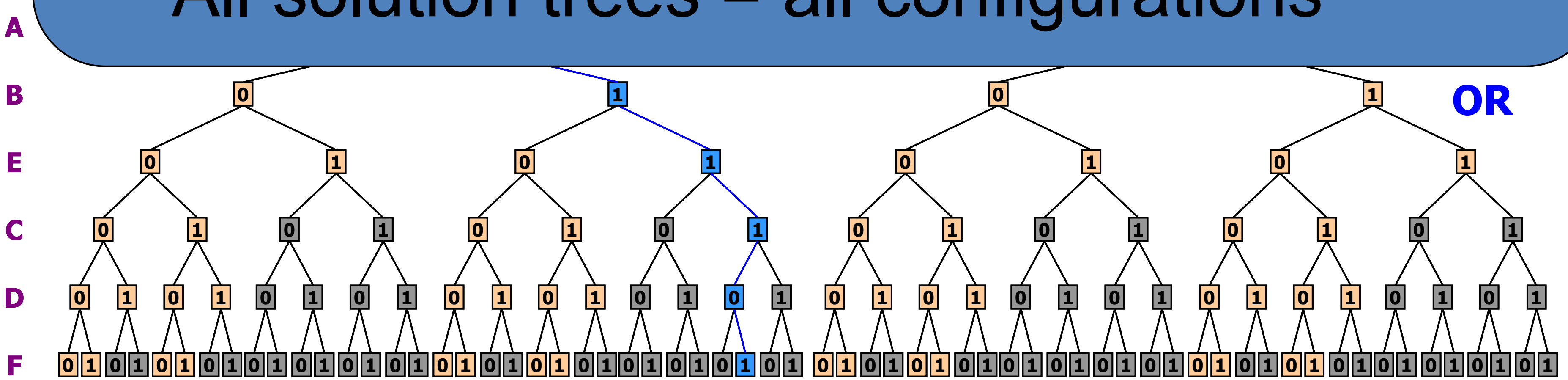


AND/OR vs. OR





- *Size of tree O(nk^h)*
- *Can be traversed in*
- *Time O(nk^h),* Space O(n)
- All solution trees = all configurations



Arc weights
Cost of a solution tree
The value function

Arc Weights for AND/OR Trees

$P(E \mid A,B)$

A	B	E=0	E=1
0	0	.4	.6
0	1	.5	.5
1	0	.7	.3
1	1	.2	.8

Evidence: E=0

$P(B \mid A)$

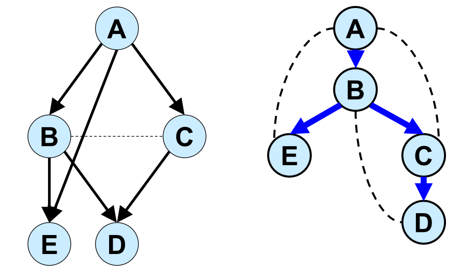
A	B=0	B=1
0	.4	.6
1	.1	.9

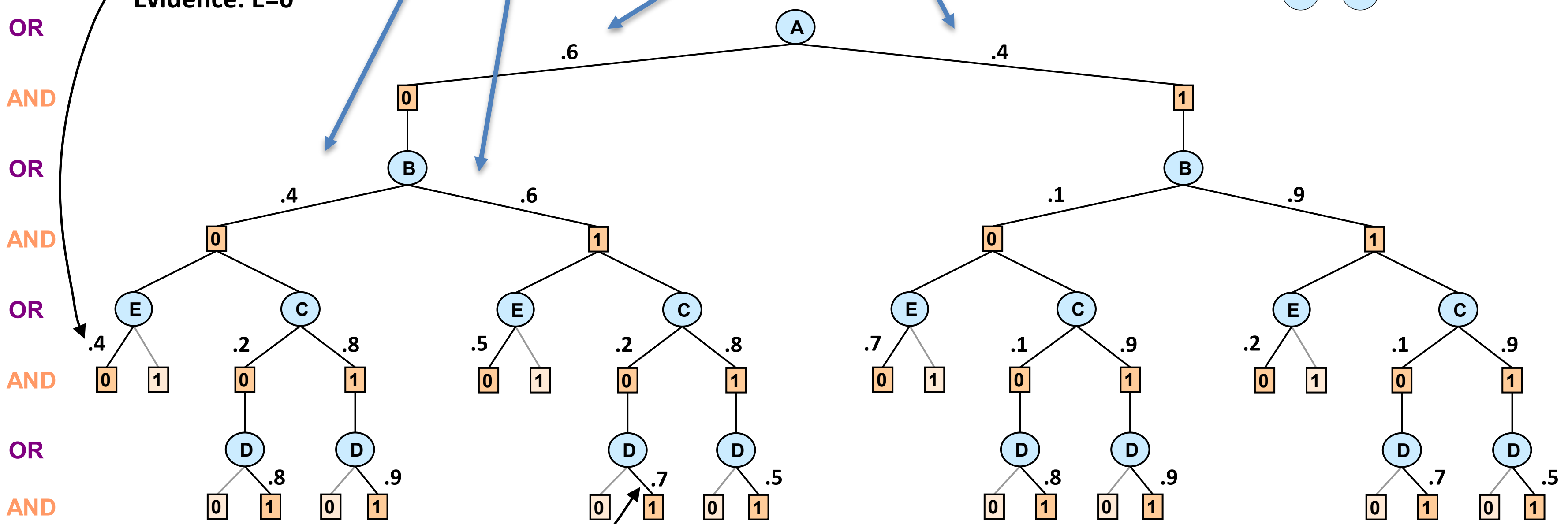
$P(C \mid A)$

A	C=0	C=1
0	.2	.8
1	.7	.3

$P(A)$

A	P(A)
0	.6
1	.4





$P(D \mid B,C)$

B	C	D=0	D=1
0	0	.2	.8
0	1	.1	.9
1	0	.3	.7
1	1	.5	.5

Evidence: D=1

OR to AND arc weight <X,x> is the product of factors that all their arguments are just assigned at AND node X=x but not before

Cost of a Solution Tree

$P(E \mid A,B)$

A	B	E=0	E=1
0	0	.4	.6
0	1	.5	.5
1	0	.7	.3
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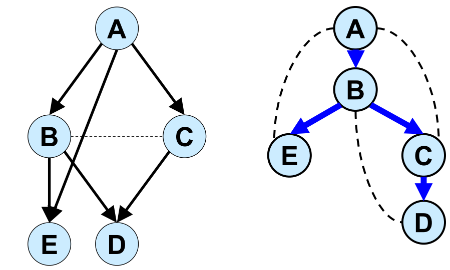

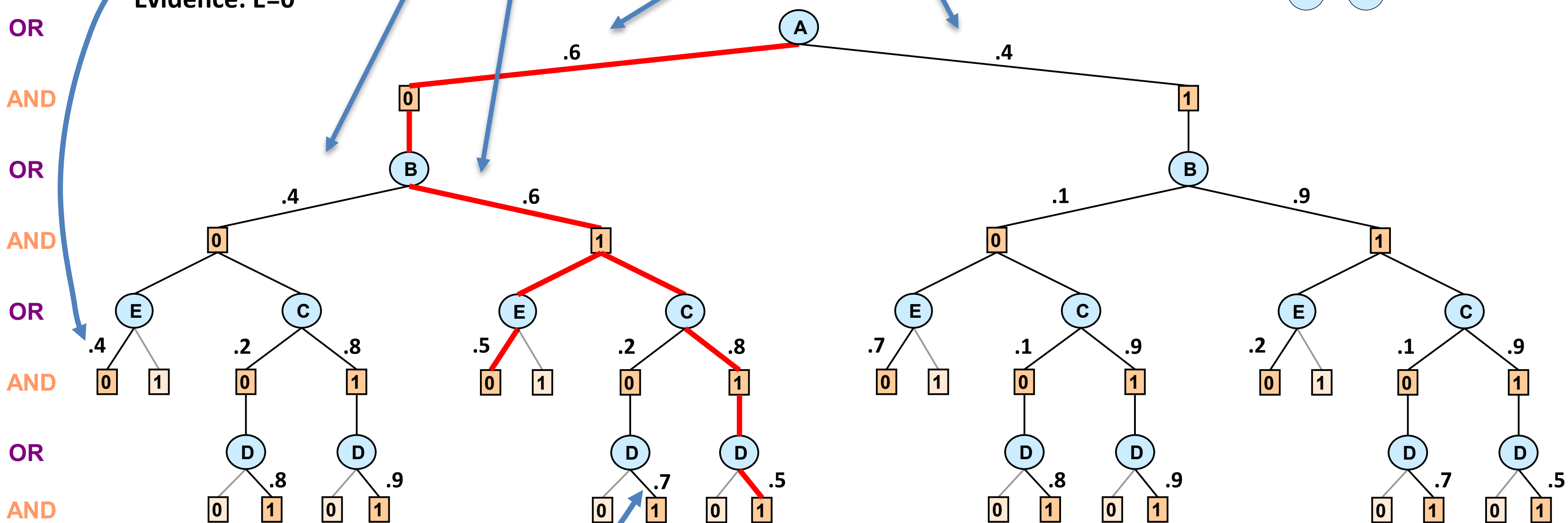

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B	C	D=0	D=1
0	0	.2	.8
0	1	.1	.9
1	0	.3	.7
1	1	.5	.5

Evidence: D=1

A solution tree includes the root and has a single child for any OR node, and all children of any of its AND nodes

Cost of the solution tree: the product of weights on its arcs

Cost of (A=0,B=1,C=1,D=1,E=0) = 0.6 · 0.6 · 0.5 · 0.8 · 0.5 = 0.0720

The Value Function for (Probability of Evidence)

$P(E \mid A, B)$

A	B	E=0	E=1
0	0	.4	.6
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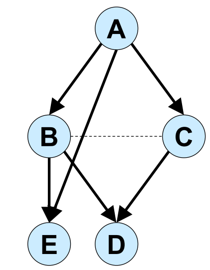
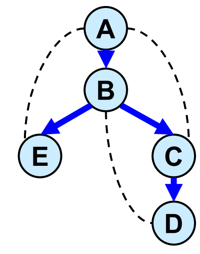
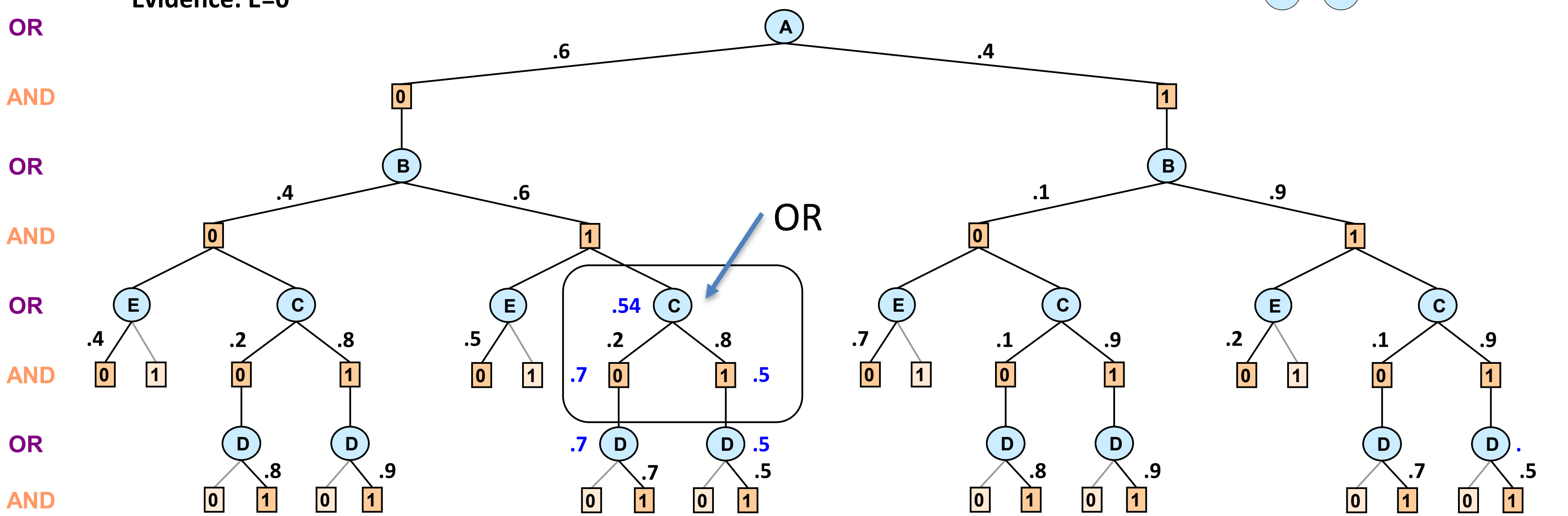




P(D=1,E=0)=?



$P(D \mid B, C)$

B	C	D=0	D=1
0	0	.2	.8
0	1	.1	.9
1	0	.3	.7
1	1	.5	.5

Evidence: D=1

Value of node = updated belief for sub-problem below

AND node: product $\prod_{n' \in children(n)} v(n')$

OR node: Marginalization by summation $\sum_{n' \in children(n)} w(n,n')\, v(n')$

The Value Function (Probability of Evidence)

$P(E \mid A,B)$

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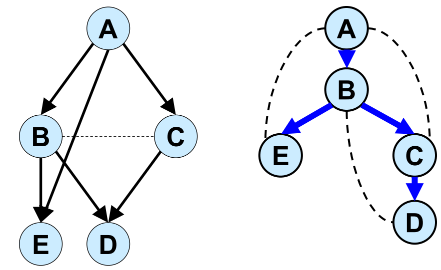

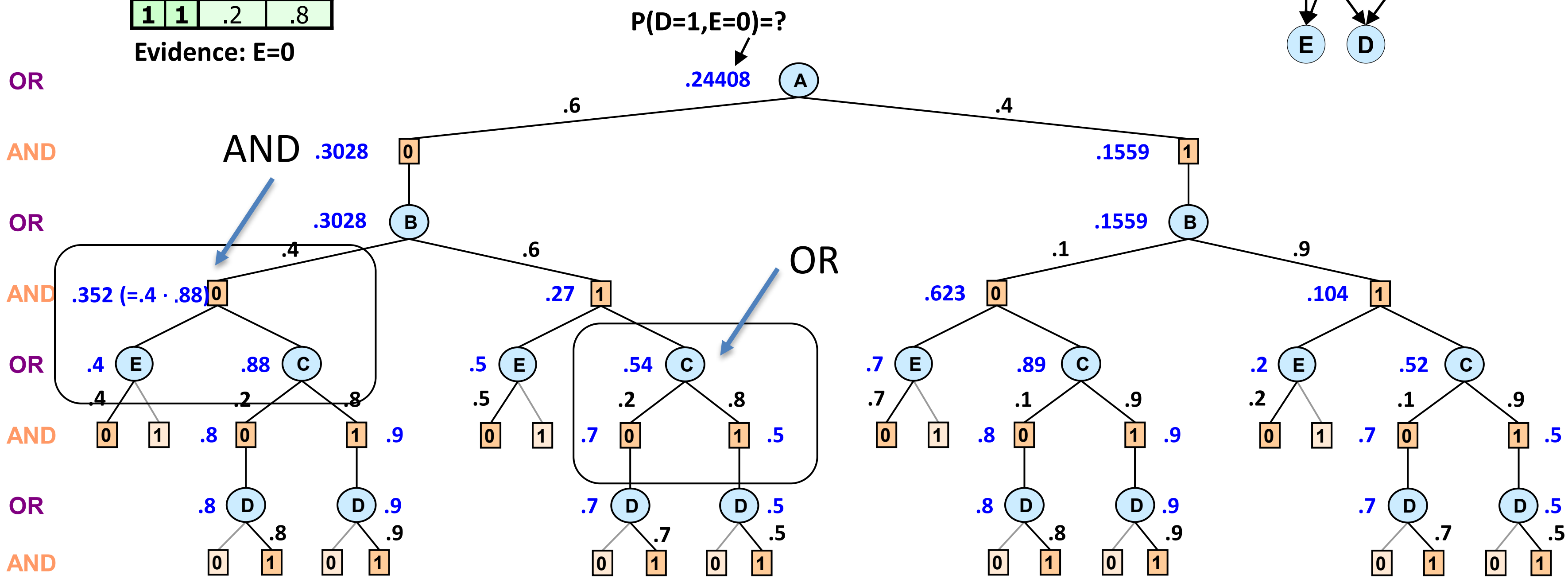

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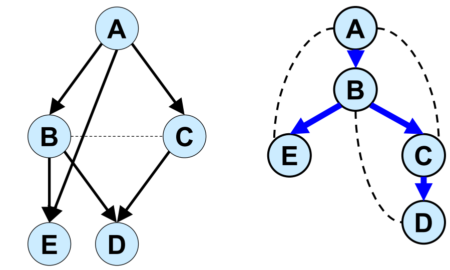
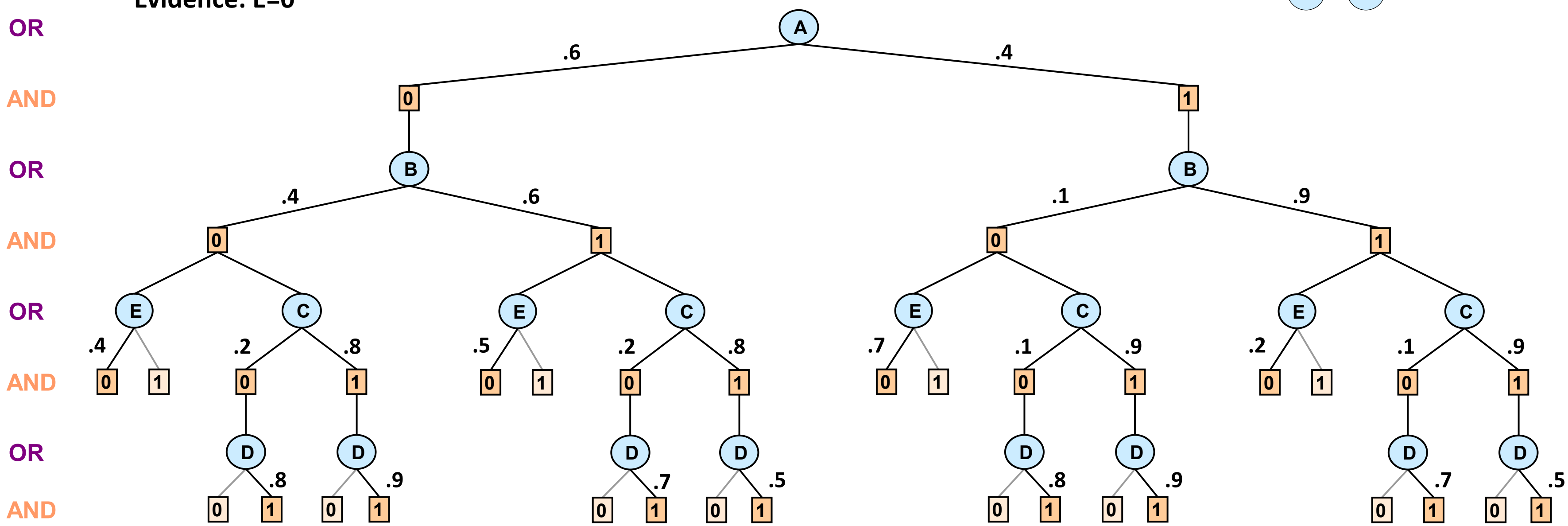




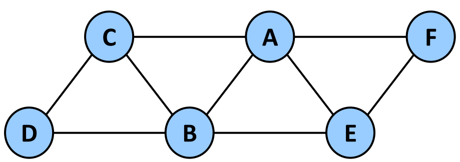
$P(D \mid B,C)$

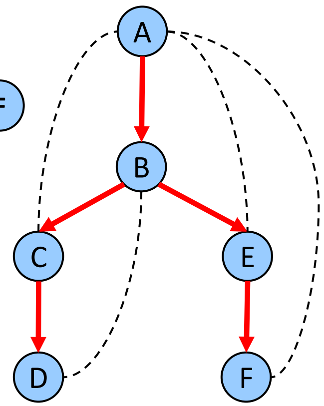
B	C	D=0	D=1
0	0	.2	.8
0	1	.1	.9
1	0	.3	.7
1	1	.5	.5

Evidence: D=1

- V(n) is dictated by the query of interest
- V(n) the value of the sub-problem represented by T(n)
- For sum-inference it is the probability mess below n
- Can be computed recursively based on child values.

The Value Function for Optimization





A	B	f_1
0	0	2
0	1	0
1	0	1
1	1	4

A	C	f_2
0	0	3
0	1	0
1	0	0
1	1	1

A	E	f_3
0	0	0
0	1	3
1	0	2
1	1	0

A	F	f_4
0	0	2
0	1	0
1	0	0
1	1	2

B	C	f_5
0	0	0
0	1	1
1	0	2
1	1	4

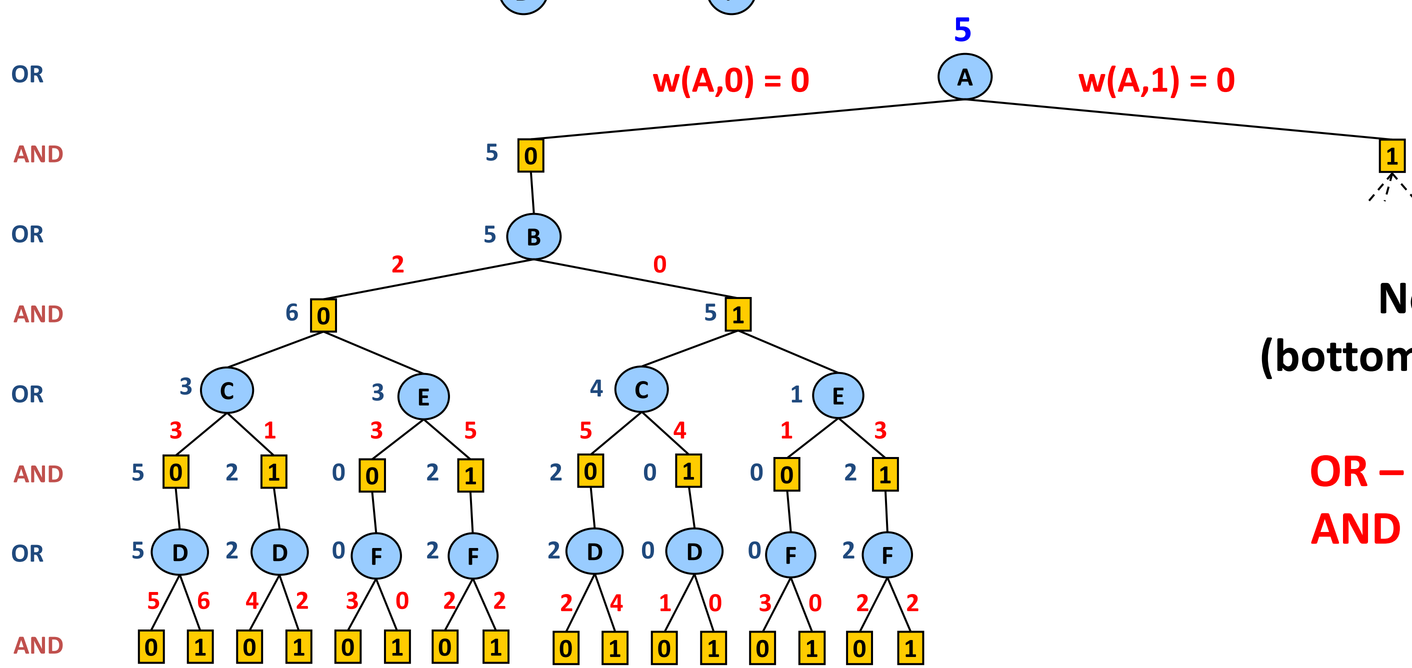
B	D	f_6
0	0	4
0	1	2
1	0	1
1	1	0

B	E	f_7
0	0	3
0	1	2
1	0	1
1	1	0

C	D	f_8
0	0	1
0	1	4
1	0	0
1	1	0

E	F	f_9
0	0	1
0	1	0
1	0	0
1	1	2

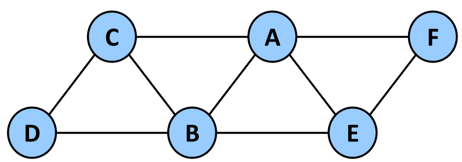
Objective function: $F^* = \min_x \sum_\alpha f_\alpha(x_\alpha)$

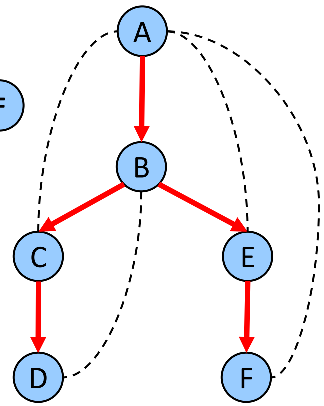


Node Value (bottom-up evaluation)

OR – minimization
AND – summation

The Value Function for Optimization





A	B	f_1
0	0	2
0	1	0
1	0	1
1	1	4

A	C	f_2
0	0	3
0	1	0
1	0	0
1	1	1

A	E	f_3
0	0	0
0	1	3
1	0	2
1	1	0

A	F	f_4
0	0	2
0	1	0
1	0	0
1	1	2

B	C	f_5
0	0	0
0	1	1
1	0	2
1	1	4

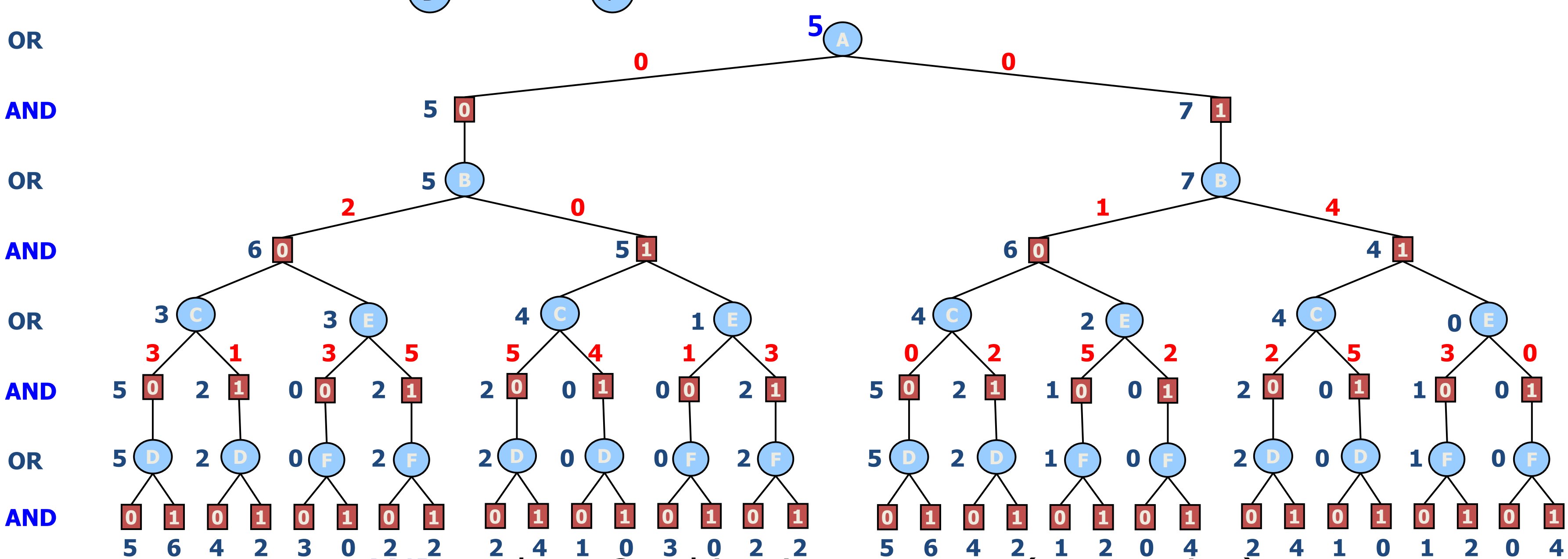
B	D	f_6
0	0	4
0	1	2
1	0	1
1	1	0

B	E	f_7
0	0	3
0	1	2
1	0	1
1	1	0

C	D	f_8
0	0	1
0	1	4
1	0	0
1	1	0

E	F	f_9
0	0	1
0	1	0
1	0	0
1	1	2

Objective function: $F^* = \min_x \sum_\alpha f_\alpha(x_\alpha)$

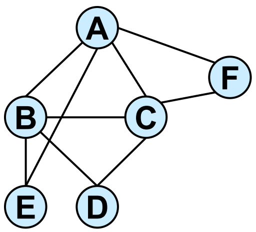


AND node = Combination operator (summation)

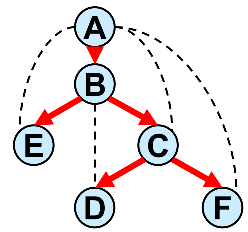
OR node = Marginalization operator (minimization)

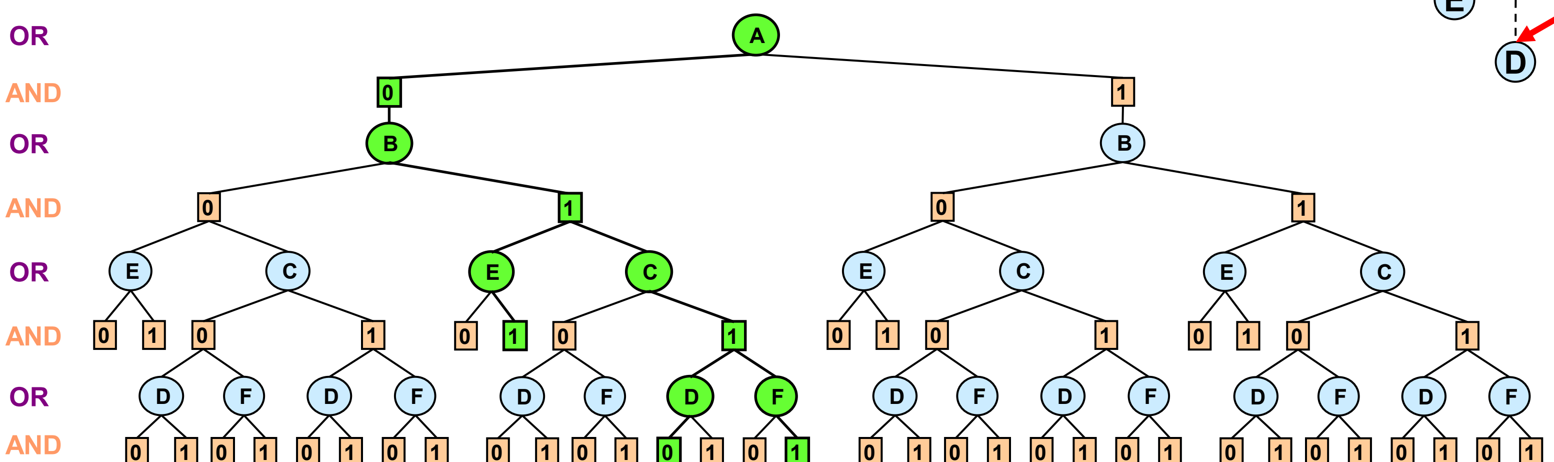
Summary: AND/OR Search Trees for GMs

- **The AND/OR search tree of R relative to a pseudo-tree, T, has:**
  - **Alternating levels of: OR nodes (variables) and AND nodes (values)**
- **Successor function:**
  - **The successors of OR nodes X are all its consistent values along its path**
  - **The successors of AND <X,v> are all X child variables in T**
  - **Arc-weight are assigned from the model factors**



- **A solution is a consistent subtree. Its cost, the product of the weights.**
- **Query: compute the value of the root node**





Size and Traversal of AND/OR Search Tree

	AND/OR tree	OR tree
Space	O(n)	O(n)
Size= Time	$O(n\ k^h)$ $O(n\ k^{w^* \log n})$ (Freuder & Quinn85), (Collin, Dechter & Katz91), (Bayardo & Miranker95), (Darwiche01)	$O(k^n)$

k = domain size
h= height of pseudo-tree
n = number of variables
w*= treewidth

$$h \leq w^* \log n$$

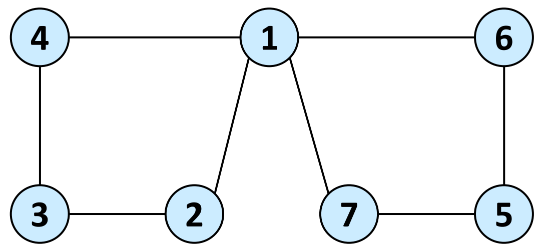
AND/OR vs. OR Spaces

width	height	OR space		AND/OR space		
		Time (sec.)	Nodes	Time (sec.)	AND nodes	OR nodes
5	10	3.15	2,097,150	0.03	**10,494**	5,247
4	9	3.13	2,097,150	0.01	**5,102**	2,551
5	10	3.12	2,097,150	0.03	**8,926**	4,463
4	10	3.12	2,097,150	0.02	**7,806**	3,903
5	13	3.11	2,097,150	0.10	**36,510**	18,255

Random graphs with 20 nodes, 20 edges and 2 values per node

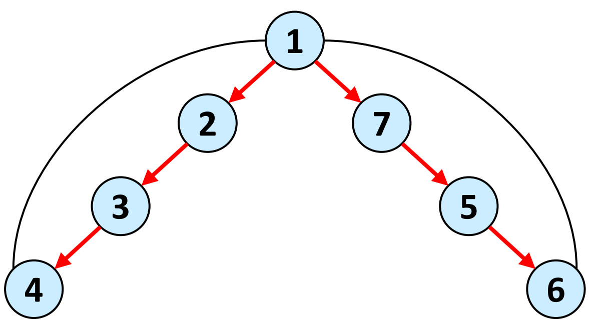
Pseudo Trees

A psedo-tree of a graph is a tree spanning its nodes, where all arcs in the graph not in the tree are back-arcs

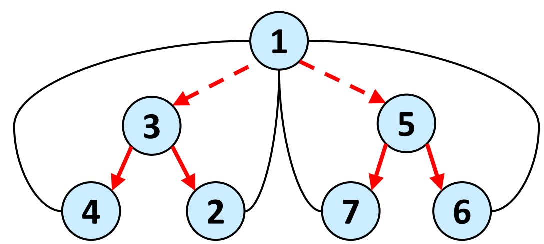


(a) Graph

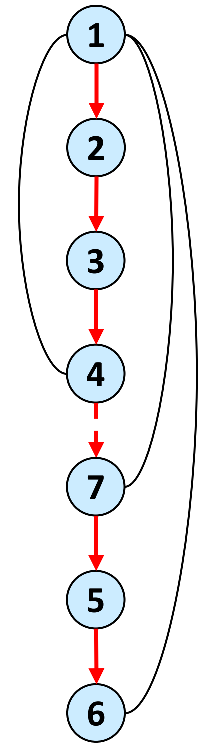
$$h \leq w^* \log n$$



(b) DFS tree
height=3

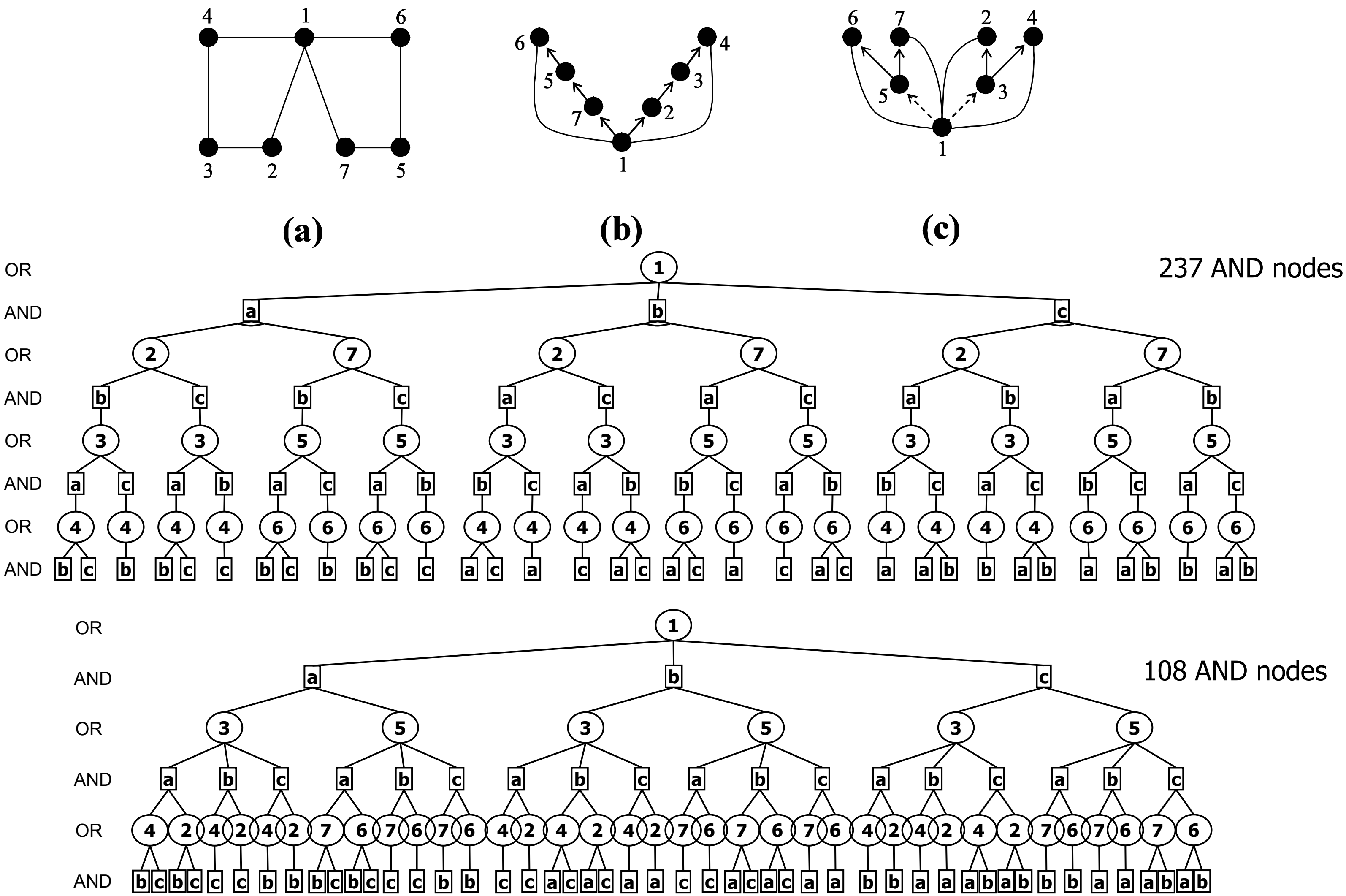


(c) Pseudo tree
height=2



(d) Chain
height=6

From DFS-Trees to Pseudo-Trees



AND/OR Search-Tree Properties

(k **= domain size,** ***h*** **= pseudo-tree height.** ***n*** **= number of variables)**

- **Theorem:** Any AND/OR search tree based on a pseudo-tree is sound and complete (expresses all and only solutions)
- **Theorem:** Size of AND/OR search tree is $O(n\ k^h)$

  Size of OR search tree is $O(k^n)$
- **Theorem:** Size of AND/OR search tree can be bounded by $O(\exp(w^* \log n))$
- When the pseudo-tree is a chain we get an OR space

Summary: Queries and Value of Nodes

- V(n) is the value of the tree T(n) for the task:
  - Max-Inference: v(n) is the optimal solution in T(n)
  - Sum-Inference: v(n) is probability of evidence in T(n).
  - Mixed-Inference: v(n) is the marginal map in T(n).
  - Mixed-Inference: v(n) is the max-expect utility in T(n) of ID.

- Goal: compute the value of the root node recursively traversing the AND/OR tree.

Complexity of searching depth-first is

- Space: $O(n)$
- Time: $O(nk^h)$
- Time: $O(k^{w*logn})$

Outline: Search

- AND/OR Search Trees
- **AND/OR Search Graphs**
- Pseudo trees generation

AND/OR Search spaces

- Basic Search (depth and Best)
- AND/OR Depth and Best Heuristic Search
- The Guiding MBE Heuristic
- Searching for Mixed tasks

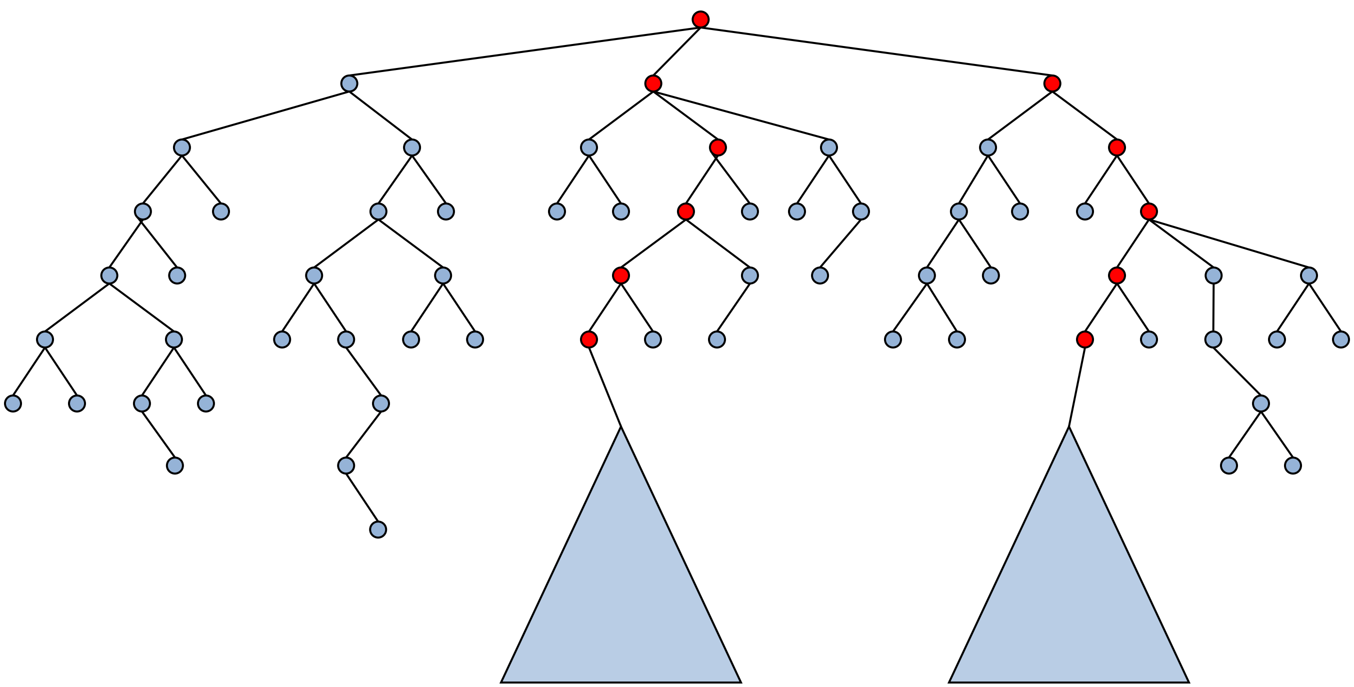
AND/OR Heuristic Search

- Hybrid of Search and Inference

Search & Inference

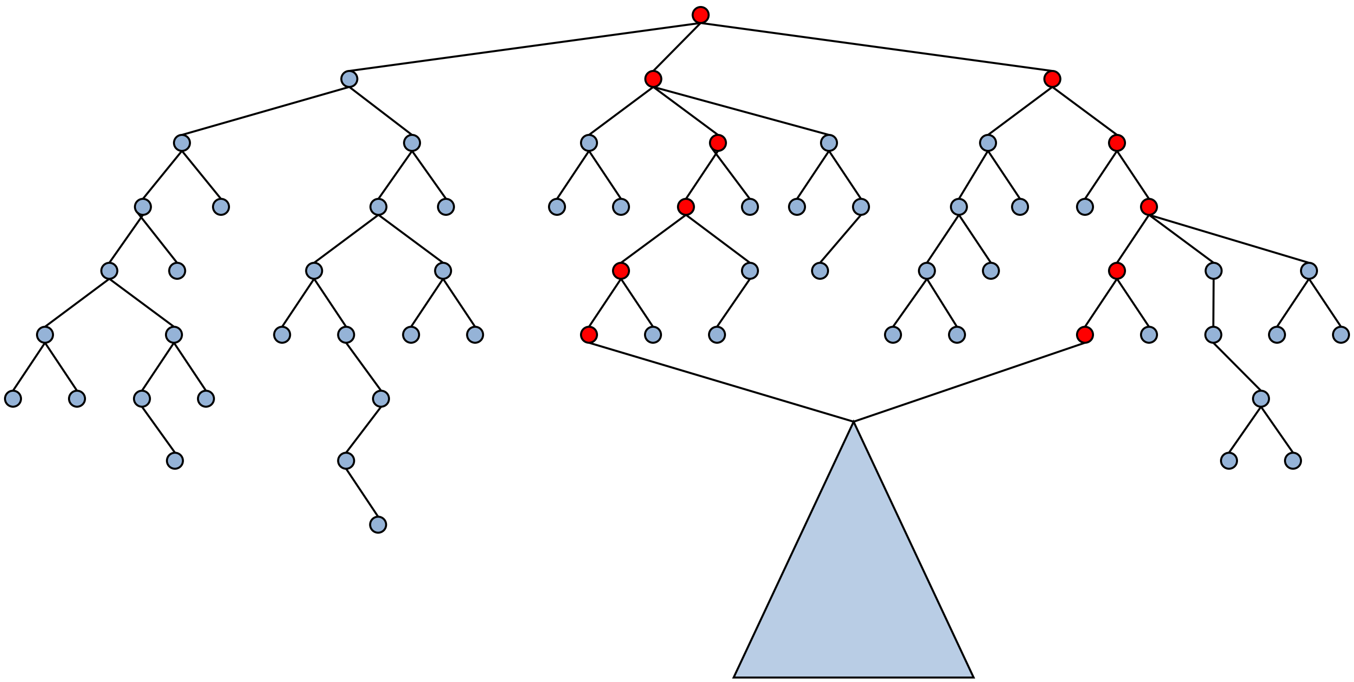
From Search Trees to Search Graphs

- Any two nodes that root identical subtrees or subgraphs can be merged

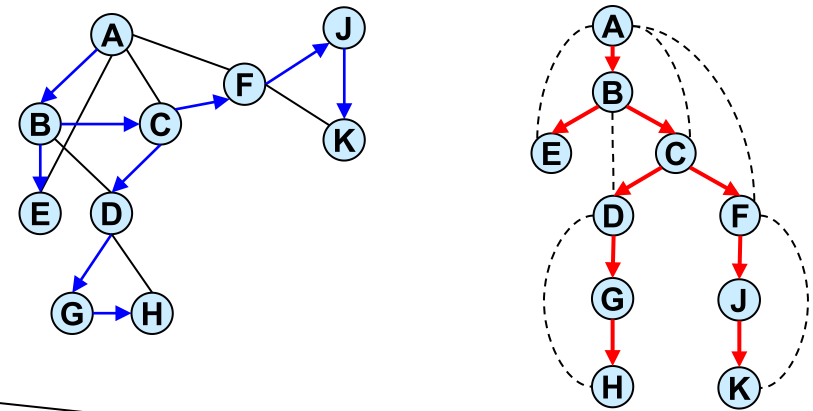


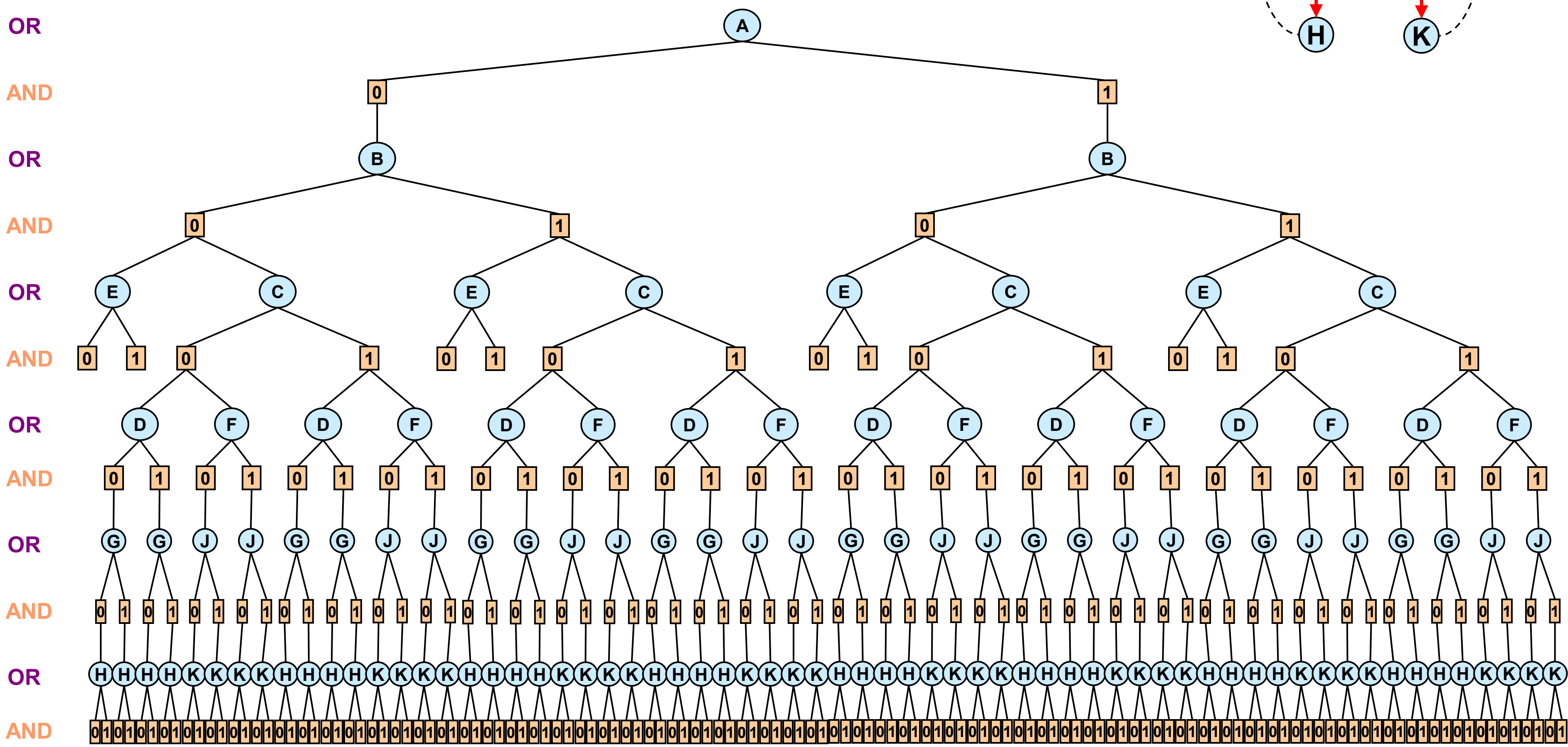
From Search Trees to Search Graphs

- Any two nodes that root identical subtrees or subgraphs can be merged



AND/OR Tree


A
B
C
D
E
F
G
H
J
K
A
B
C
D
E
F
G
H
J
K


OR
AND
OR
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OR
AND
OR
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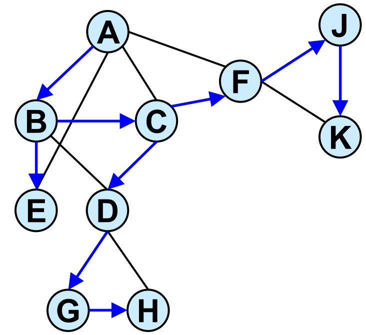
OR
AND
OR
AND

AND/OR Graph


A
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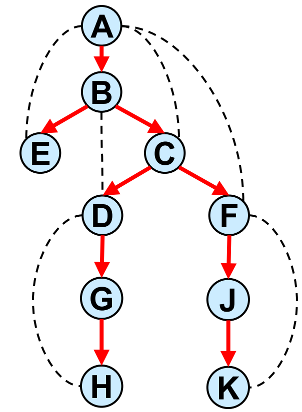
K


A
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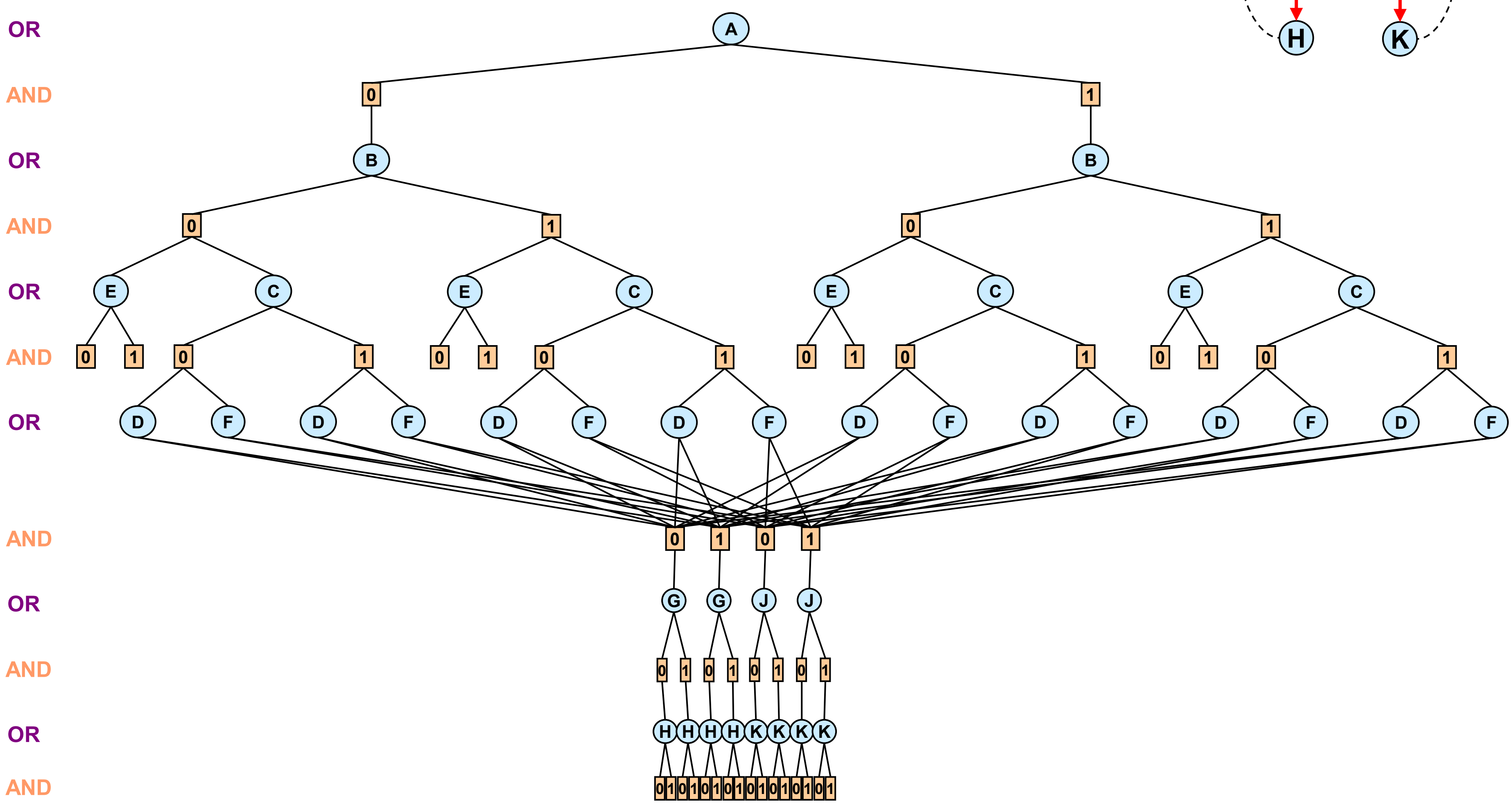
OR
AND
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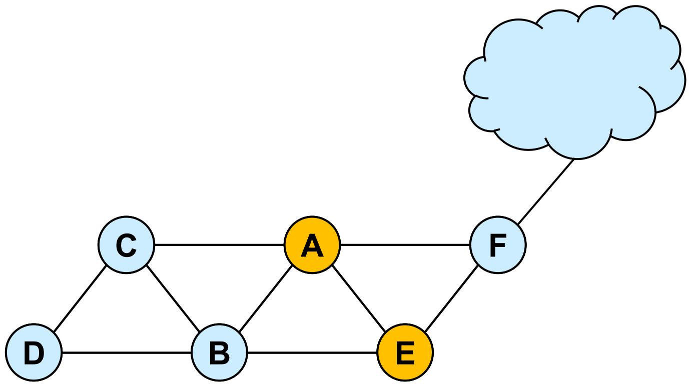
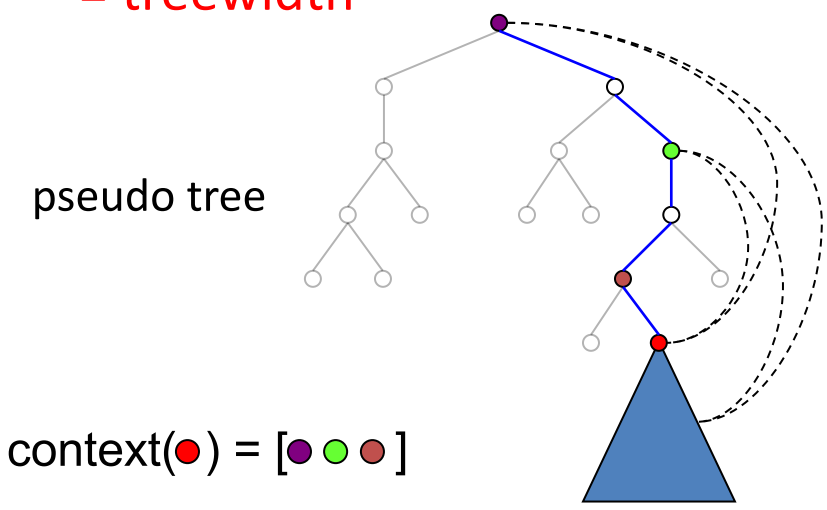
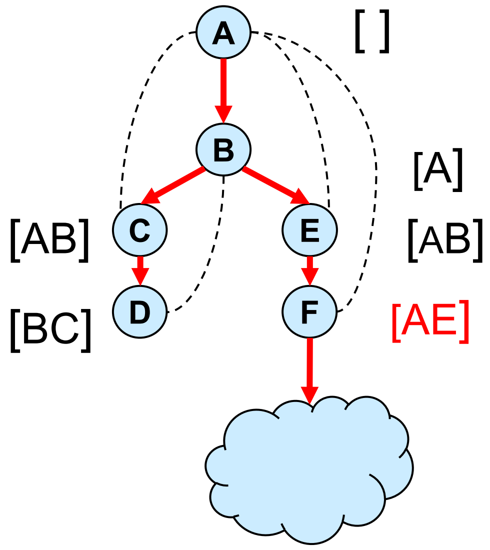
AND
OR
AND

Merging Based on Context

- context (X) = ancestors of X in pseudo tree, connected to X, or to descendants of X
- context (X) = parents in the induced graph
- max |context| = induced width = treewidth







Context-Based Minimal AND/OR Search Graph

Definition 7.2.13 (context minimal AND/OR search graph) *The AND/OR search graph of M guided by a pseudo-tree T that is closed under context-based merge operator, is called the* context minimal *AND/OR search graph and is denoted by* $C_T(R)$.

AND/OR Tree DFS Algorithm (Value=Sum-Product)

$P(E \mid A,B)$

A	B	E=0	E=1
0	0	**.4**	.6
0	1	.5	.5
1	0	.7	.3
1	1	.2	.8

Evidence: E=0

$P(B \mid A)$

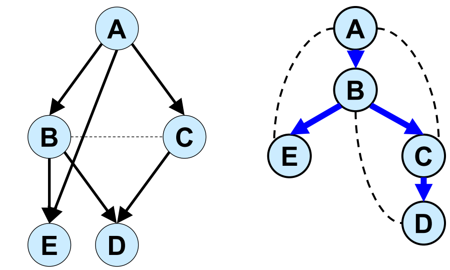
A	B=0	B=1
0	.4	.6
1	.1	.9

$P(C \mid A)$

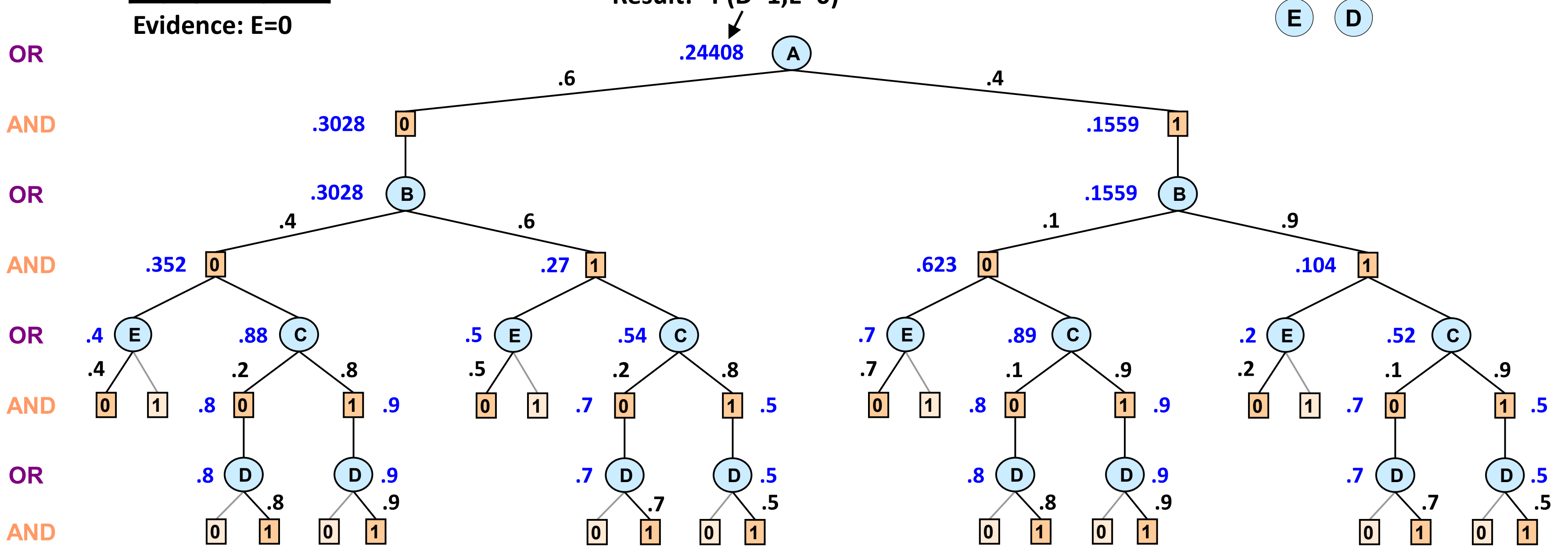
A	C=0	C=1
0	.2	.8
1	.7	.3

$P(A)$

A	P(A)
0	.6
1	.4



Result: P(D=1,E=0)



$P(D \mid B,C)$

B	C	D=0	D=1
0	0	.2	.8
0	1	.1	.9
1	0	.3	**.7**
1	1	.5	.5

Evidence: D=1

Evidenc: D=1

AND/OR Search Graph (Value=Sum-Product)

$P(E \mid A, B)$

A	B	E=0	E=1
0	0	.4	.6
0	1	.5	.5
1	0	.7	.3
1	1	.2	.8

Evidence: E=0

$P(B \mid A)$

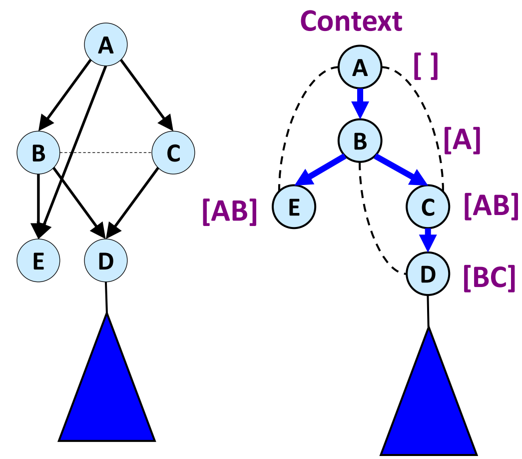
A	B=0	B=1
0	.4	.6
1	.1	.9

$P(C \mid A)$

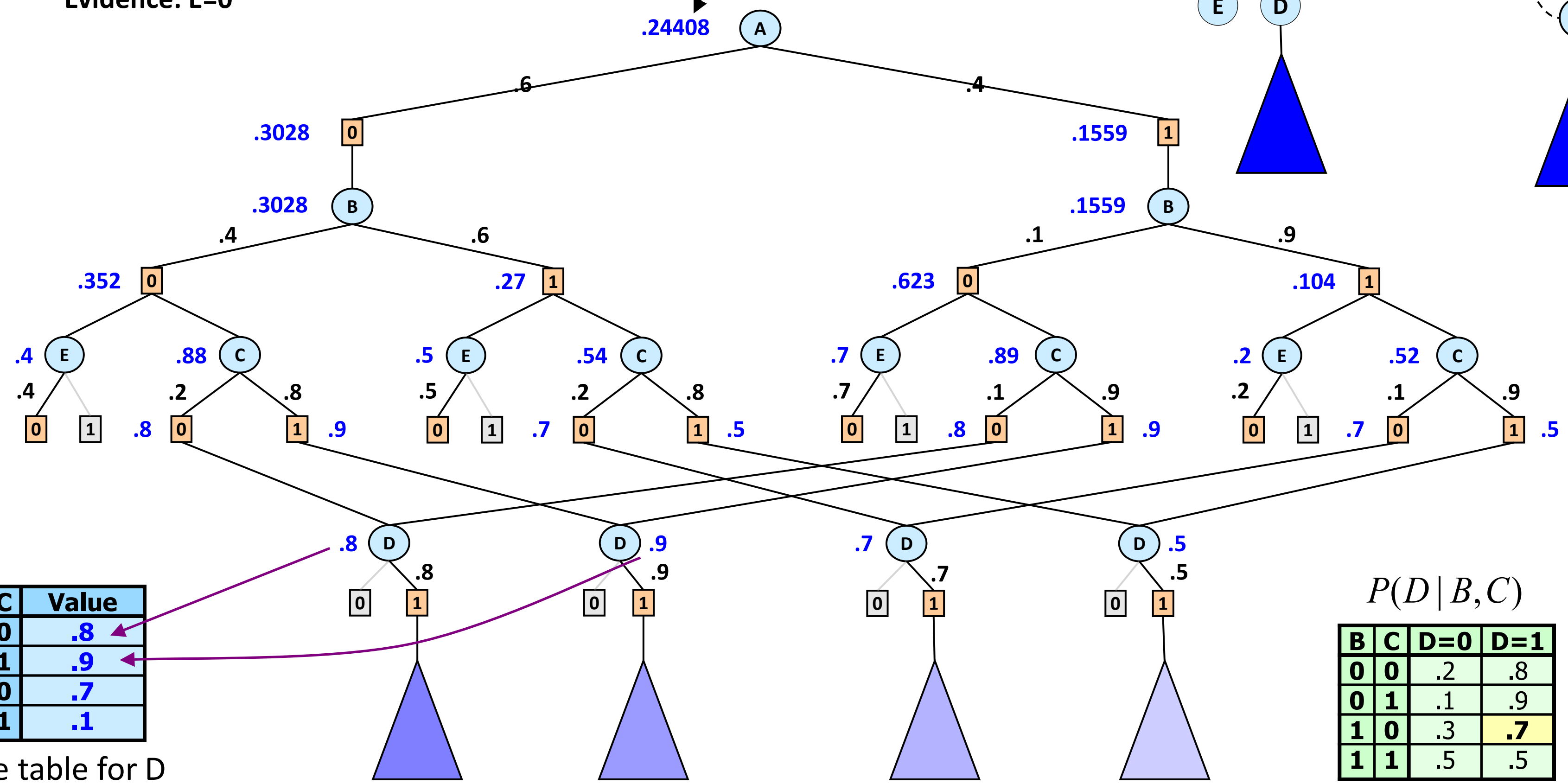
A	C=0	C=1
0	.2	.8
1	.7	.3

$P(A)$

A	P(A)
0	.6
1	.4



Result: P(D=1,E=0)



B	C	Value
0	0	.8
0	1	.9
1	0	.7
1	1	.1

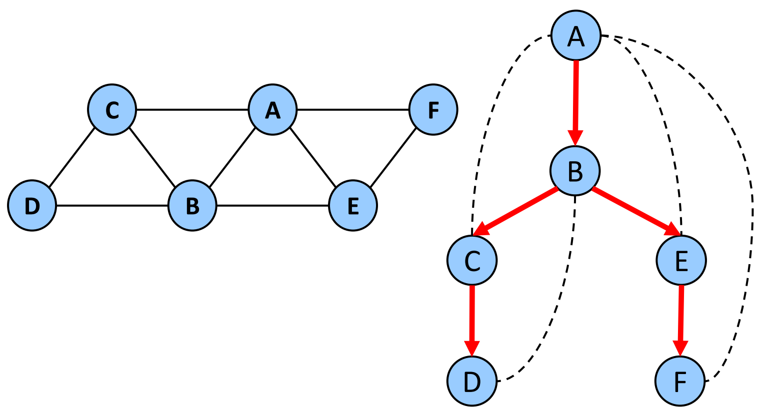
Cache table for D

$P(D \mid B, C)$

B	C	D=0	D=1
0	0	.2	.8
0	1	.1	.9
1	0	.3	.7
1	1	.5	.5

Evidence: D=1

AND/OR Search Graph (Optimization)



A	B	f_1
0	0	2
0	1	0
1	0	1
1	1	4

A	C	f_2
0	0	3
0	1	0
1	0	0
1	1	1

A	E	f_3
0	0	0
0	1	3
1	0	2
1	1	0

A	F	f_4
0	0	2
0	1	0
1	0	0
1	1	2

B	C	f_5
0	0	0
0	1	1
1	0	2
1	1	4

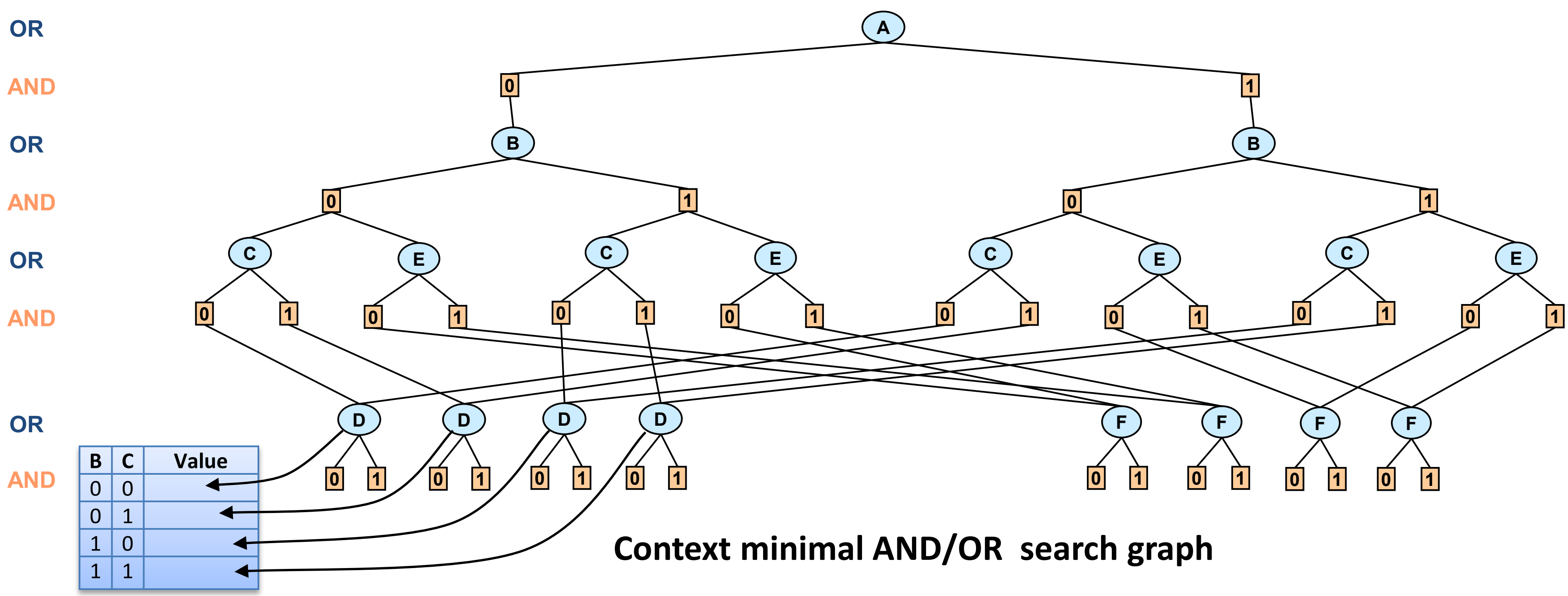
B	D	f_6
0	0	4
0	1	2
1	0	1
1	1	0

B	E	f_7
0	0	3
0	1	2
1	0	1
1	1	0

C	D	f_8
0	0	1
0	1	4
1	0	0
1	1	0

E	F	f_9
0	0	1
0	1	0
1	0	0
1	1	2

Objective function: $F^* = \min_x \sum_\alpha f_\alpha(x_\alpha)$

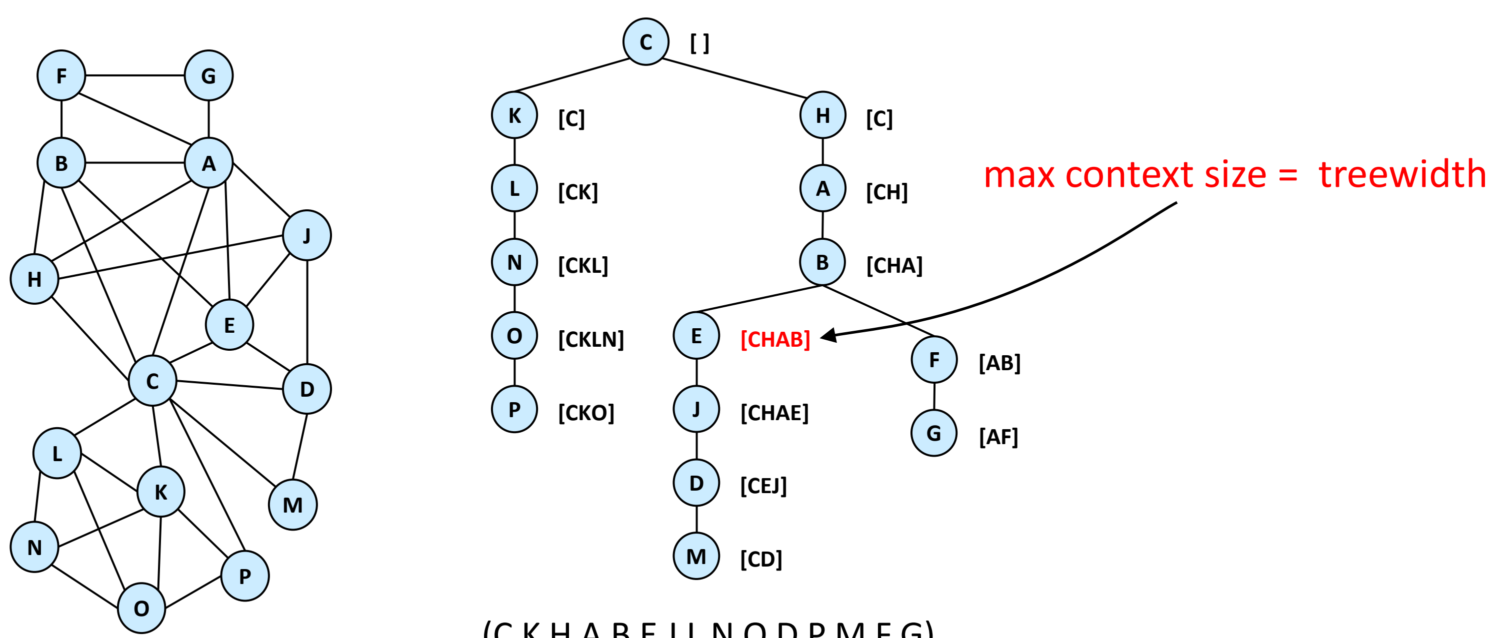


Context minimal AND/OR search graph

Cache table for **D**

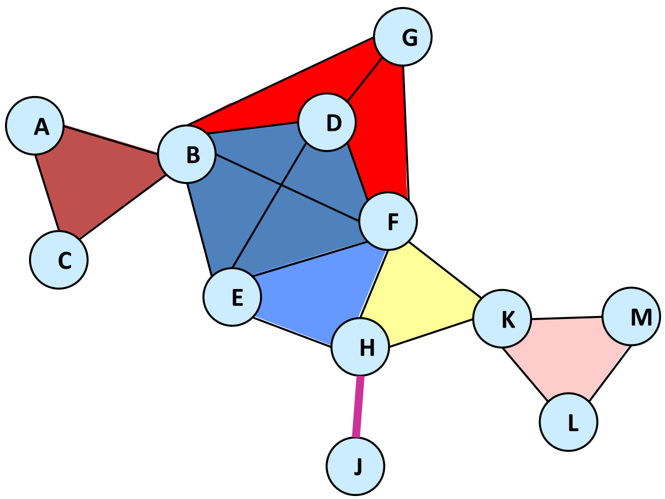
How Big Is The Context?

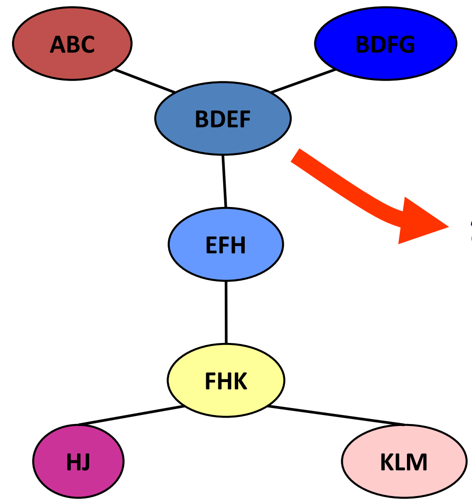
- Theorem: The maximum context-size of a pseudo-tree equals the treewidth along the pseudo tree.



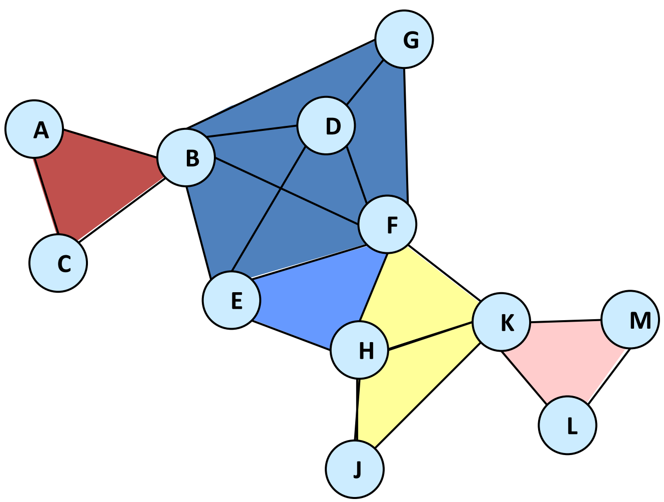
(C K H A B E J L N O D P M F G)

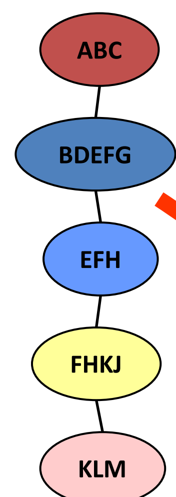
Treewidth vs. Pathwidth





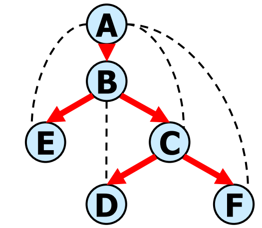
***treewidth* = 3**
= (max cluster size) - 1

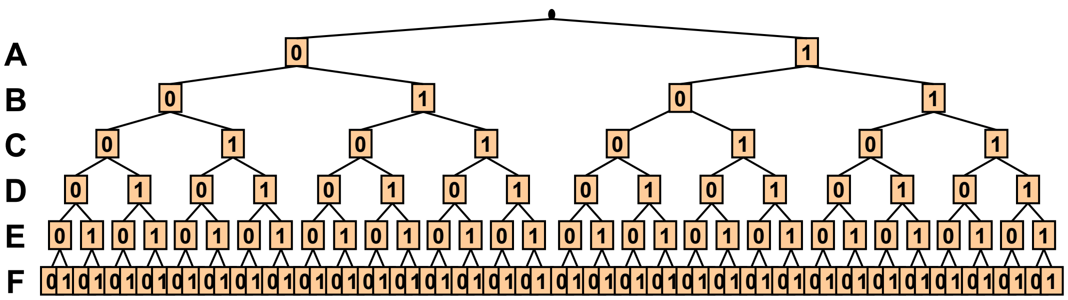




***pathwidth* = 4**
= (max cluster size) - 1

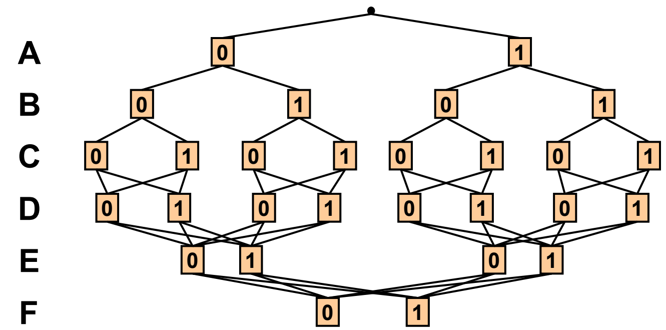
All Four Search Spaces





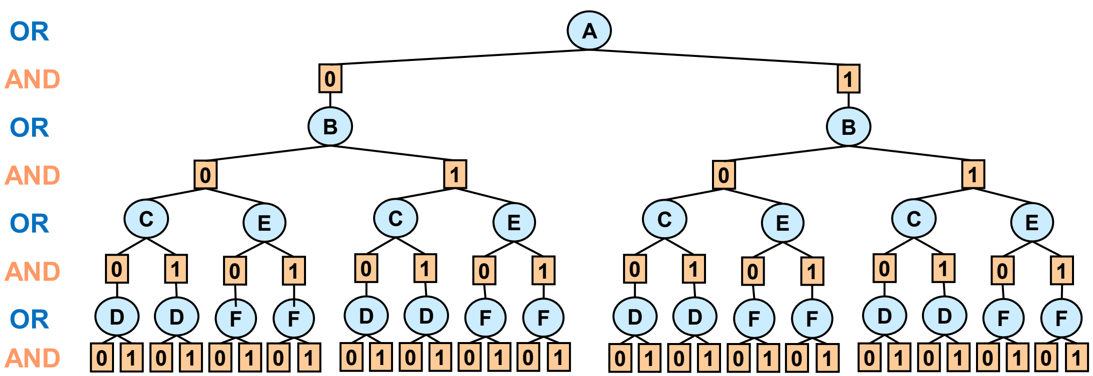
Full OR search tree

126 nodes



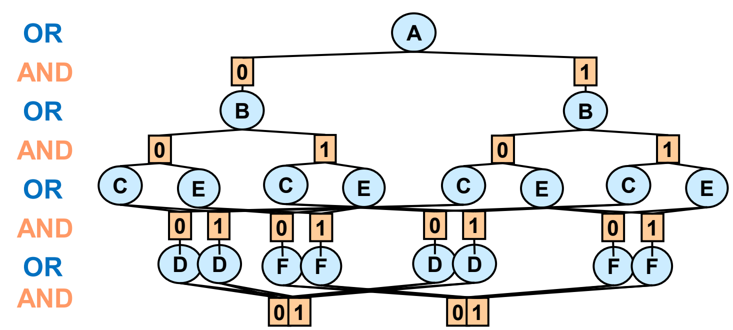
Context minimal OR search graph

28 nodes



Full AND/OR search tree

54 AND nodes



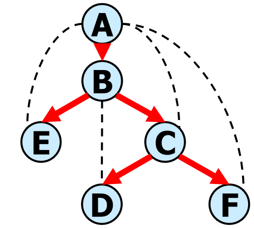
Context minimal AND/OR search graph

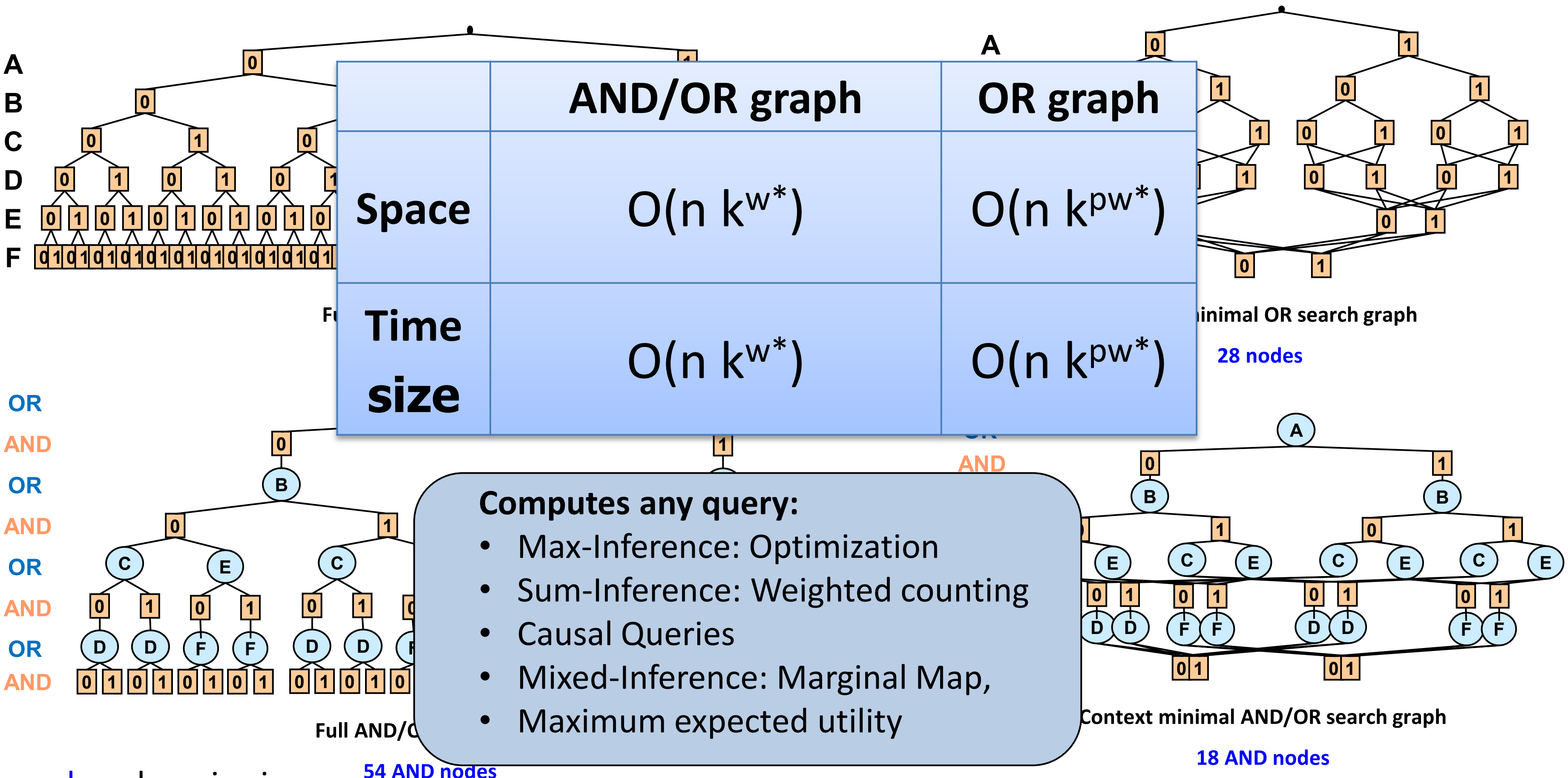
18 AND nodes

- k = domain size
- n = number of variables
- w* = treewidth
- pw* = pathwidth

Any query is best computed over the context-minimal AND/OR space

All Four Search Spaces





	AND/OR graph	OR graph
Space	$O(n\,k^{w^*})$	$O(n\,k^{pw^*})$
Time size	$O(n\,k^{w^*})$	$O(n\,k^{pw^*})$

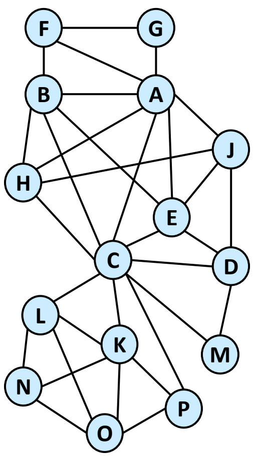
Computes any query:

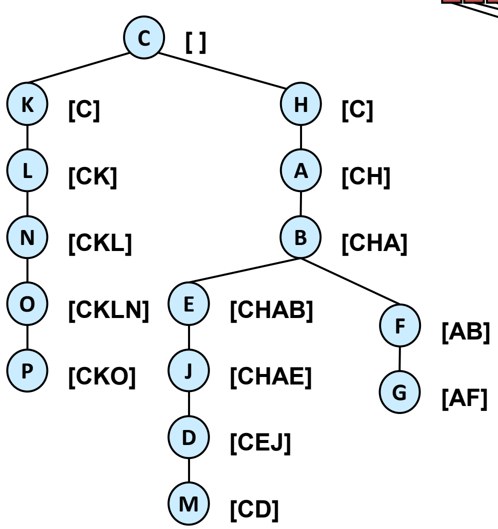
- Max-Inference: Optimization
- Sum-Inference: Weighted counting
- Causal Queries
- Mixed-Inference: Marginal Map,
- Maximum expected utility

k = domain size
n = number of variables
w* = treewidth
pw* = pathwidth

Any query is best computed over the context-minimal AND/OR space

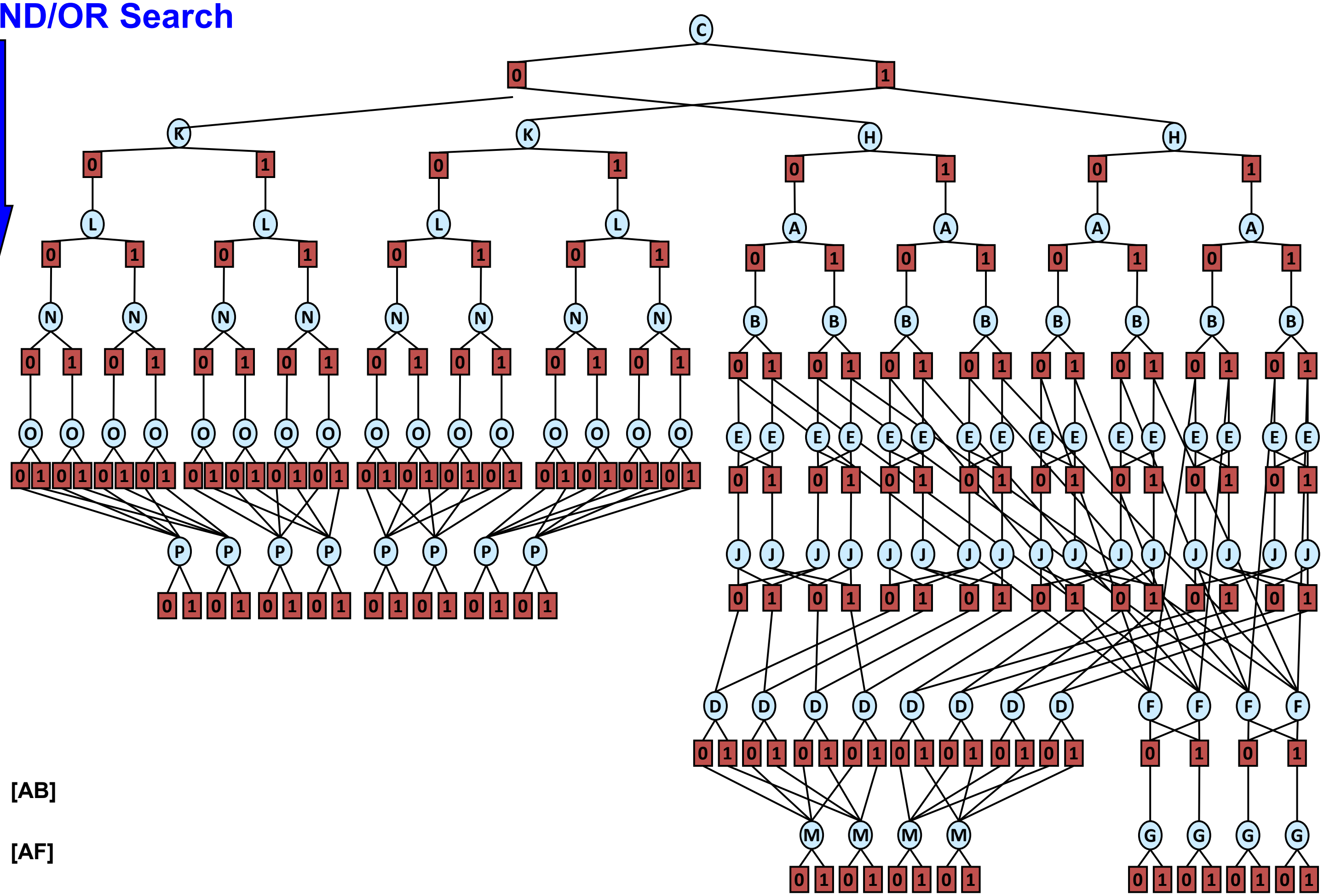
AND/OR Search and Variable Elimination



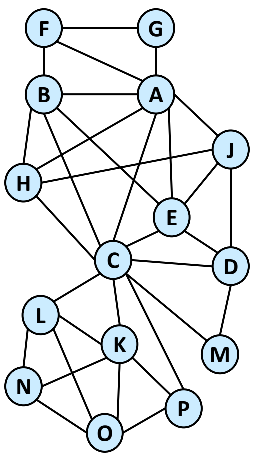


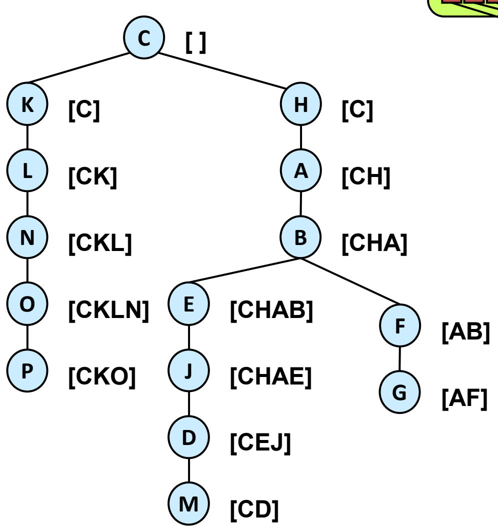
(C K H A B E J L N O D P M F G)

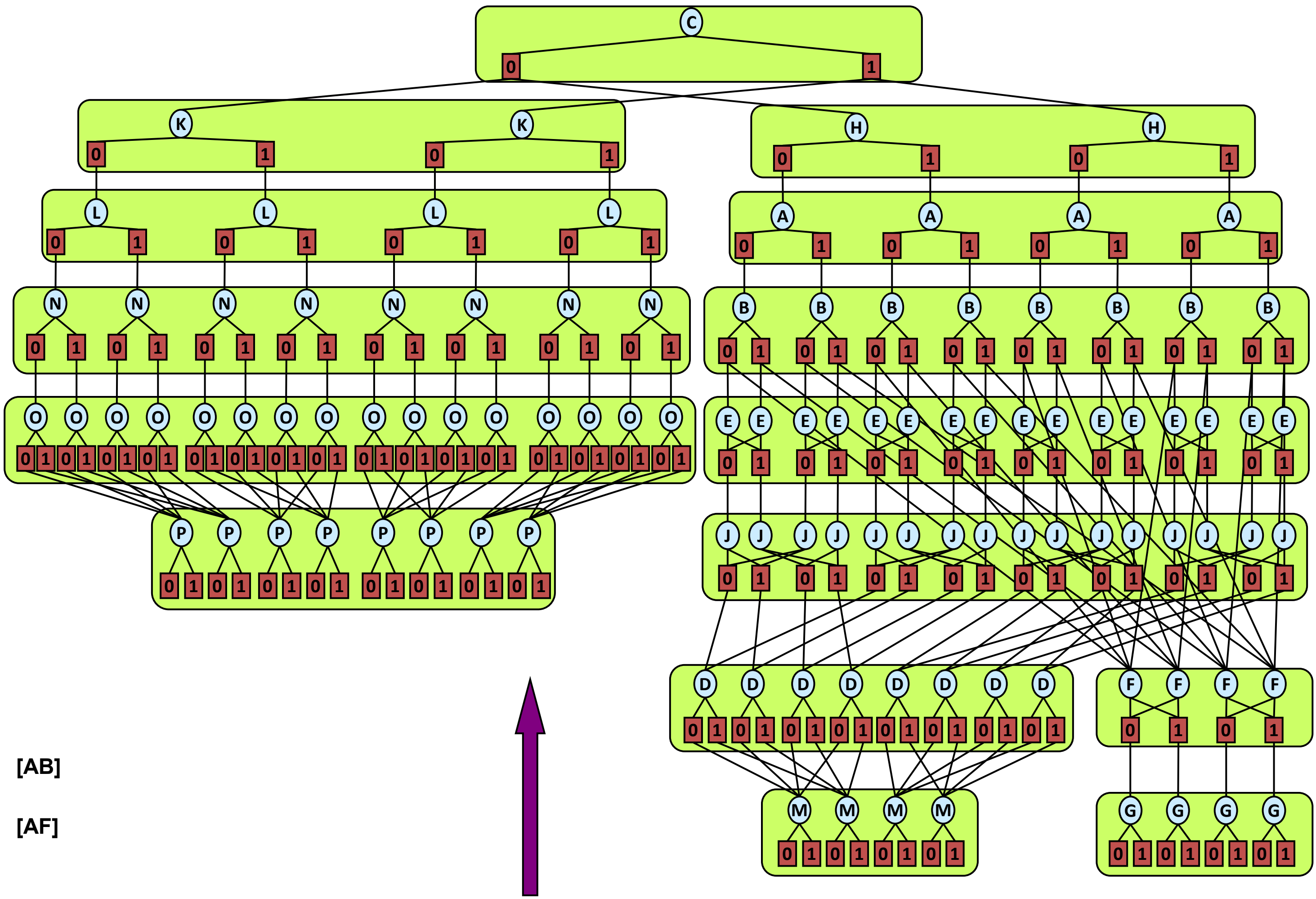
AND/OR Search



AND/OR Search and Variable Elimination



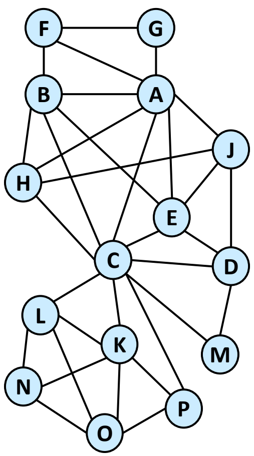


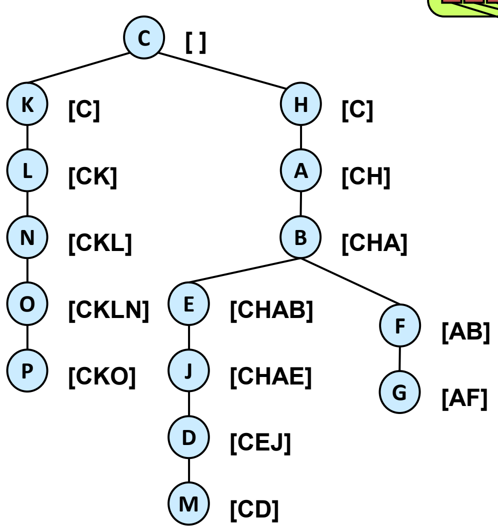


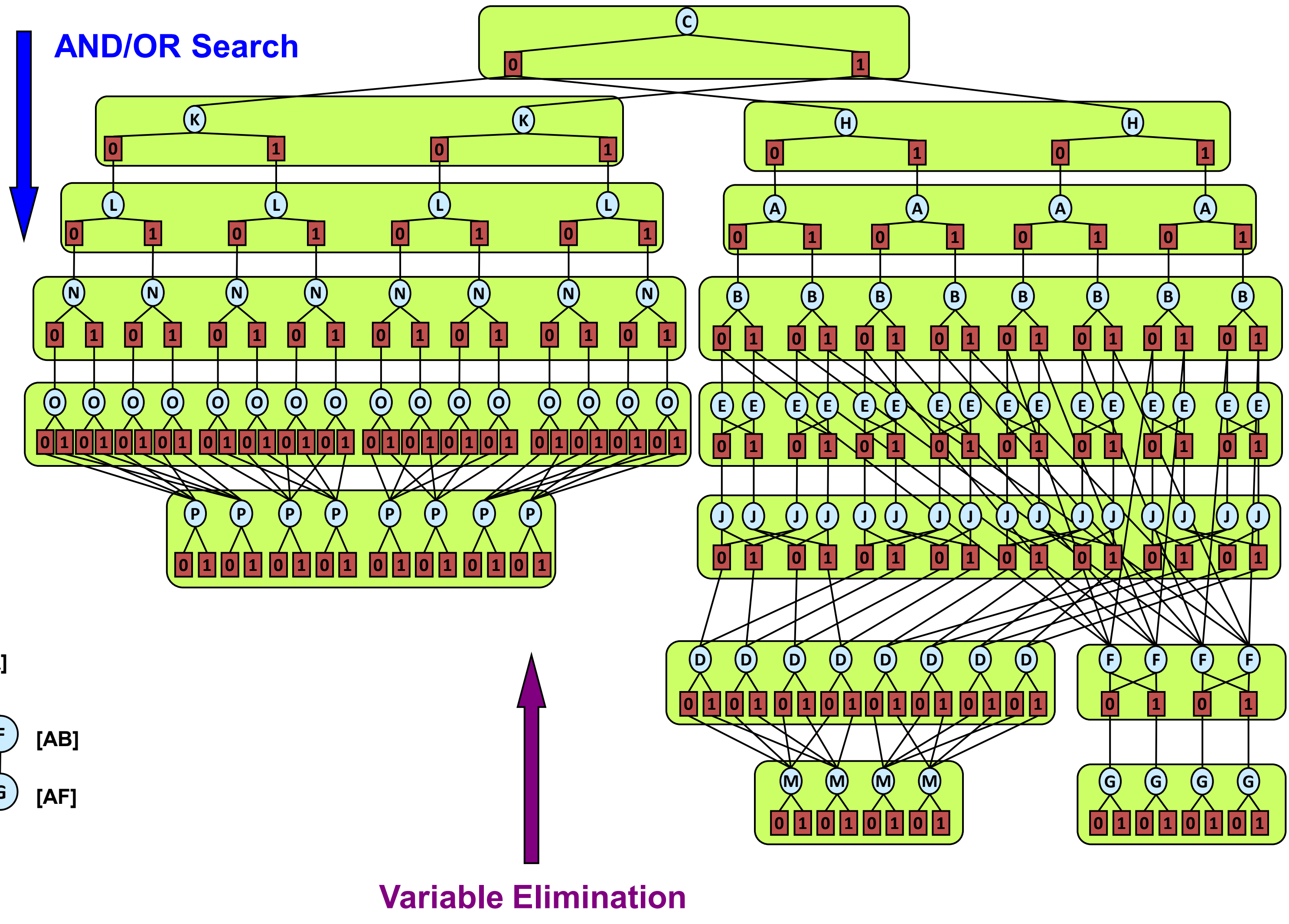
Variable Elimination

(C K H A B E J L N O D P M F G)

AND/OR Search and Variable Elimination







(C K H A B E J L N O D P M F G)

Outline: Search

- AND/OR Search Trees
- AND/OR Search Graphs
- **Generating good Pseudo trees**

AND/OR Search spaces

- Basic Search (depth and Best)
- AND/OR Depth and Best Heuristic Search
- The Guiding MBE Heuristic
- Searching for Mixed tasks

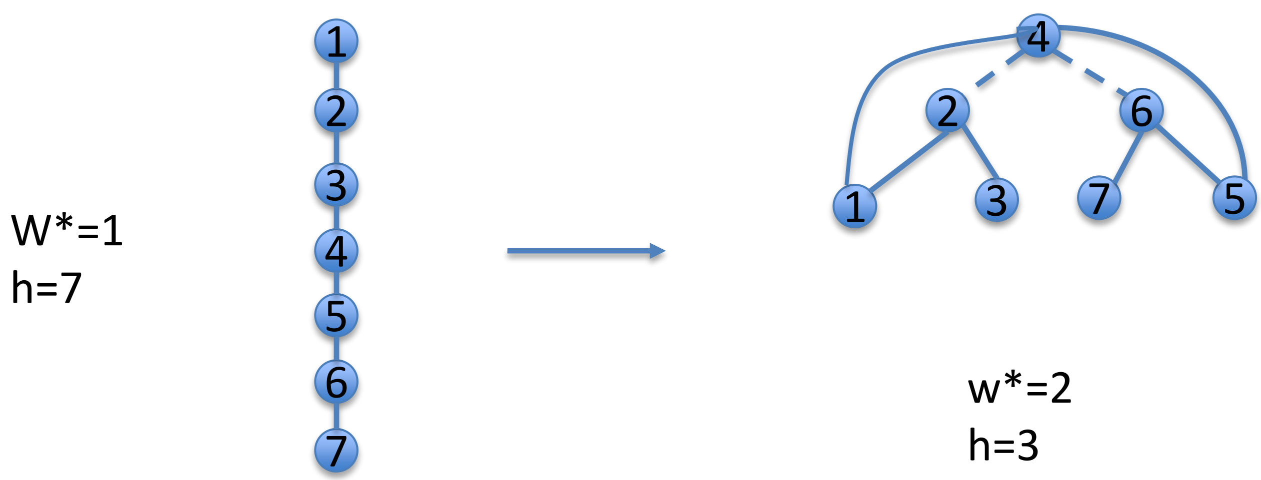
AND/OR Heuristic Search

- Hybrid of Search and Inference

Search & Inference

Finding Min-height Pseudo-Trees

- Finding a min height pseudo-tree is NP-complete, but:
- Given a tree-decomposition with treewidth w*, there exists a pseudo-tree whose height satisfies
  - h <= w* log n
- Optimality of h and w* cannot be achieved at once.



Constructing Pseudo-Trees

- **Min-Fill** [Kjaerulff, 1990]
  - Depth-first traversal of the induced graph obtained along the min-fill elimination order
  - Variables ordered according to the smallest "fill-set"

- **Hypergraph Partitioning** [Karypis and Kumar, 2000]
  - Functions are vertices in the hypergraph and variables are hyperedges
  - Recursive decomposition of the hypergraph while minimizing the separator size at each step
  - Using state-of-the-art software package hMeTiS

Quality of Pseudo-Trees

Network	hypergraph		min-fill	
	width	depth	width	depth
barley	7	**13**	7	23
diabetes	7	**16**	4	77
link	21	**40**	15	53
mildew	5	**9**	4	13
munin1	12	**17**	12	29
munin2	9	**16**	9	32
munin3	9	**15**	9	30
munin4	9	**18**	9	30
water	11	**16**	10	15
pigs	11	**20**	11	26

Bayesian Networks Repository

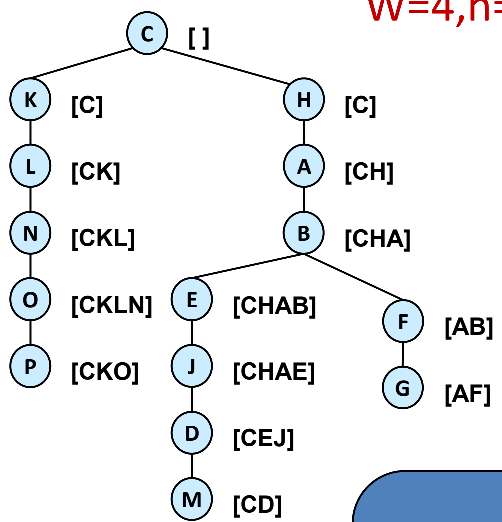
Network	hypergraph		min-fill	
	width	depth	width	depth
spot5	47	152	**39**	204
spot28	108	138	**79**	199
spot29	16	23	**14**	42
spot42	36	48	**33**	87
spot54	12	16	**11**	33
spot404	19	26	**19**	42
spot408	47	52	**35**	97
spot503	11	20	**9**	39
spot505	29	42	**23**	74
spot507	70	122	**59**	160

SPOT5 Benchmarks

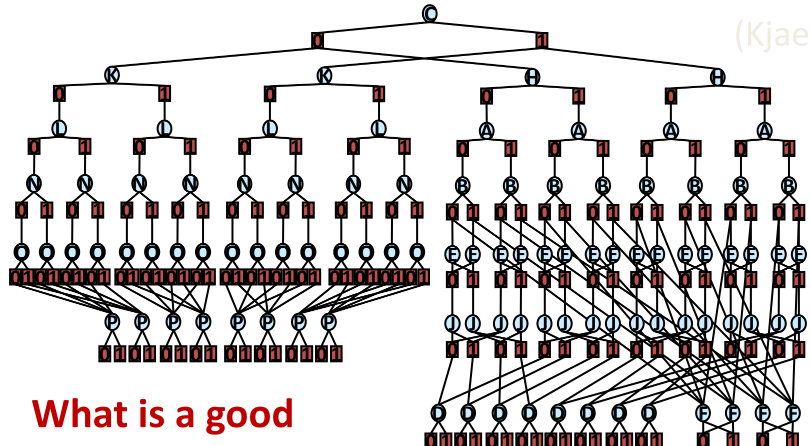
For more see [Dechter 2003]

The Impact of the Pseudo-Tree

W=4,h=8

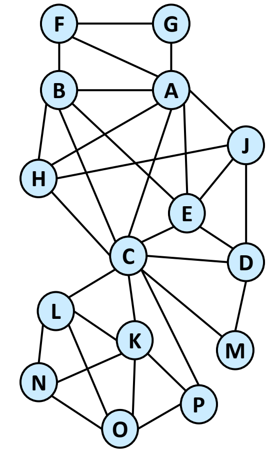


(C K H A B E J L N O



What is a good

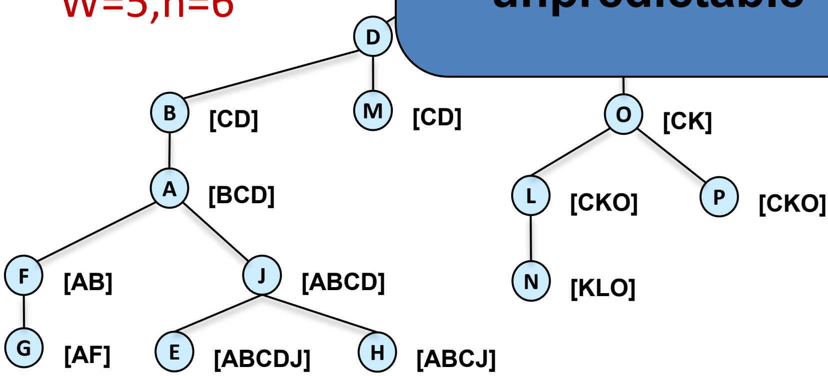
Min-Fill
(Kjaerulff90)

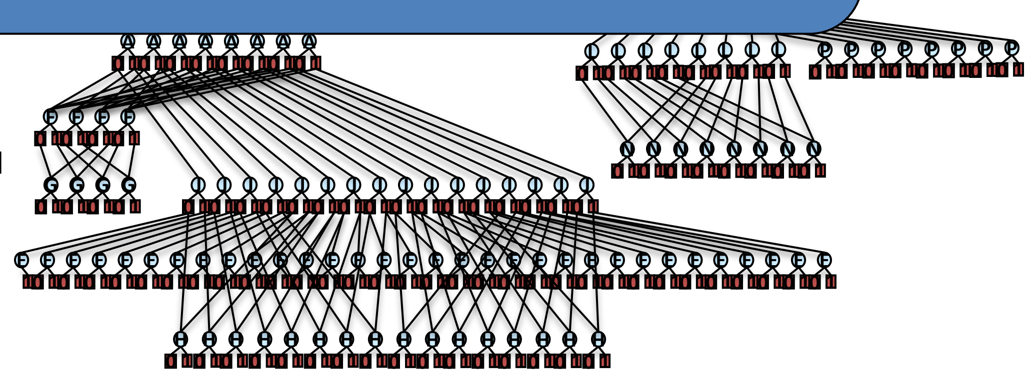


graph
oning
is)

- ***Choose pseudo-tree with a minimal search graph***
- **But determinism and pruning for optimization is unpredictable**

W=5,h=6





(C D K B A O M L N P J H E F G)

Outline: Search

- AND/OR Search Trees
- AND/OR Search Graphs
- Pseudo trees generation

AND/OR Search spaces

- **Basic Brute-Force and Heuristic Search**
- AND/OR Depth and Best Heuristic Search
- The Guiding MBE Heuristic
- Searching for Mixed tasks

AND/OR Heuristic Search

- Hybrid of Search and Inference

Search & Inference

Probabilistic Reasoning Problems

- Exact Inference by elimination or search
- Complexity:

Causal effects

Max-Inference:	$f(x^*) = \max_x \prod_\alpha f_\alpha(x_\alpha)$
Sum-Inference:	$Z = \sum_x \prod_\alpha f_\alpha(x_\alpha)$
Mixed-Inference (MMAP):	$f_M(x_M^*) = \max_{x_M} \sum_{x_S} \prod_\alpha f_\alpha(x_\alpha)$
Mixed-Inference (MEU):	$\text{MEU} = \max \sum (\prod P_i) \times (\sum r_i)$

$e^{\text{tree-width}}$

Harder

- **All solved by AND/OR Depth-first search,**
  - **Linear memory, exp(h) time or**
  - **exp(w*) memory and time**
- **But, we can do better by:**
  - **Pruning while searching**
  - **Generating upper and lower bounds anytime**

AND/OR Tree DFS Algorithm (Belief Updating)

$P(E \mid A, B)$

A	B	E=0	E=1
0	0	.4	.6
0	1	.5	.5
1	0	.7	.3
1	1	.2	.8

Evidence: E=0

$P(B \mid A)$

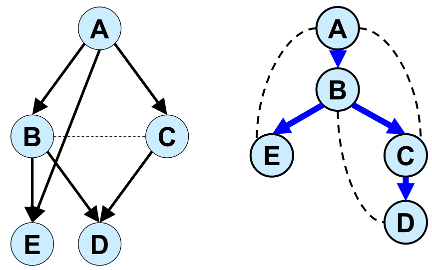
A	B=0	B=1
0	.4	.6
1	.1	.9

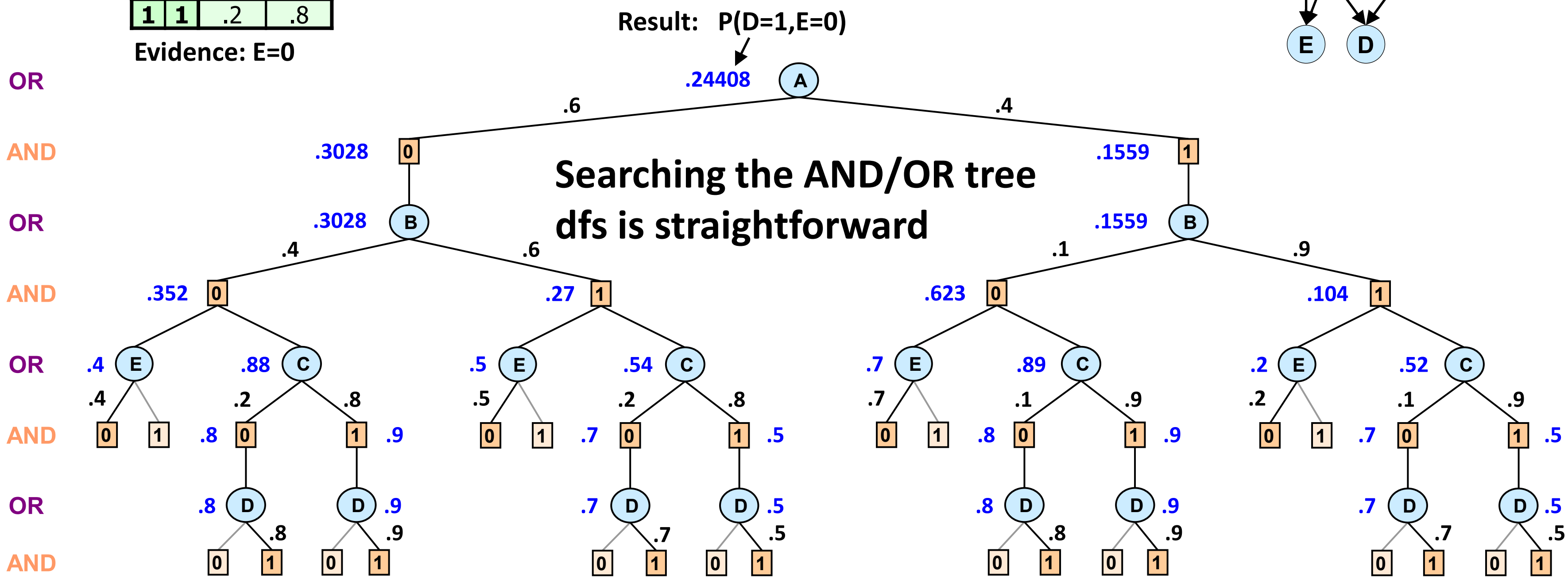
$P(C \mid A)$

A	C=0	C=1
0	.2	.8
1	.7	.3

$P(A)$

A	P(A)
0	.6
1	.4





Searching the AND/OR tree dfs is straightforward

$P(D \mid B, C)$

B	C	D=0	D=1
0	0	.2	.8
0	1	.1	.9
1	0	.3	.7
1	1	.5	.5

Evidence: D=1

OR node: Marginalization operator (summation)

AND node: Combination operator (product)

Value of node = updated belief for sub-problem below

AND/OR Graph DFS Algorithm (Belief Updating)

$P(E \mid A,B)$

A	B	E=0	E=1
0	0	.4	.6
0	1	.5	.5
1	0	.7	.3
1	1	.2	.8

Evidence: E=0

$P(B \mid A)$

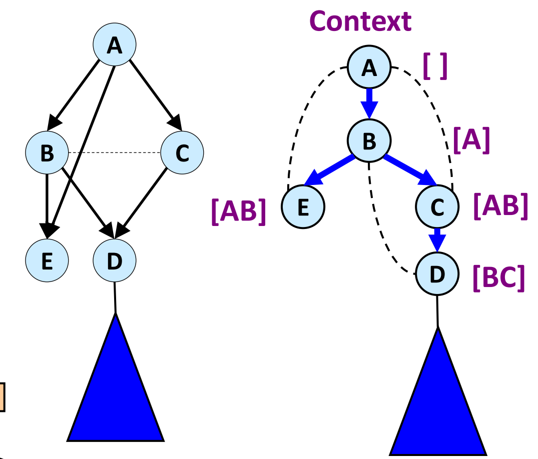
A	B=0	B=1
0	.4	.6
1	.1	.9

$P(C \mid A)$

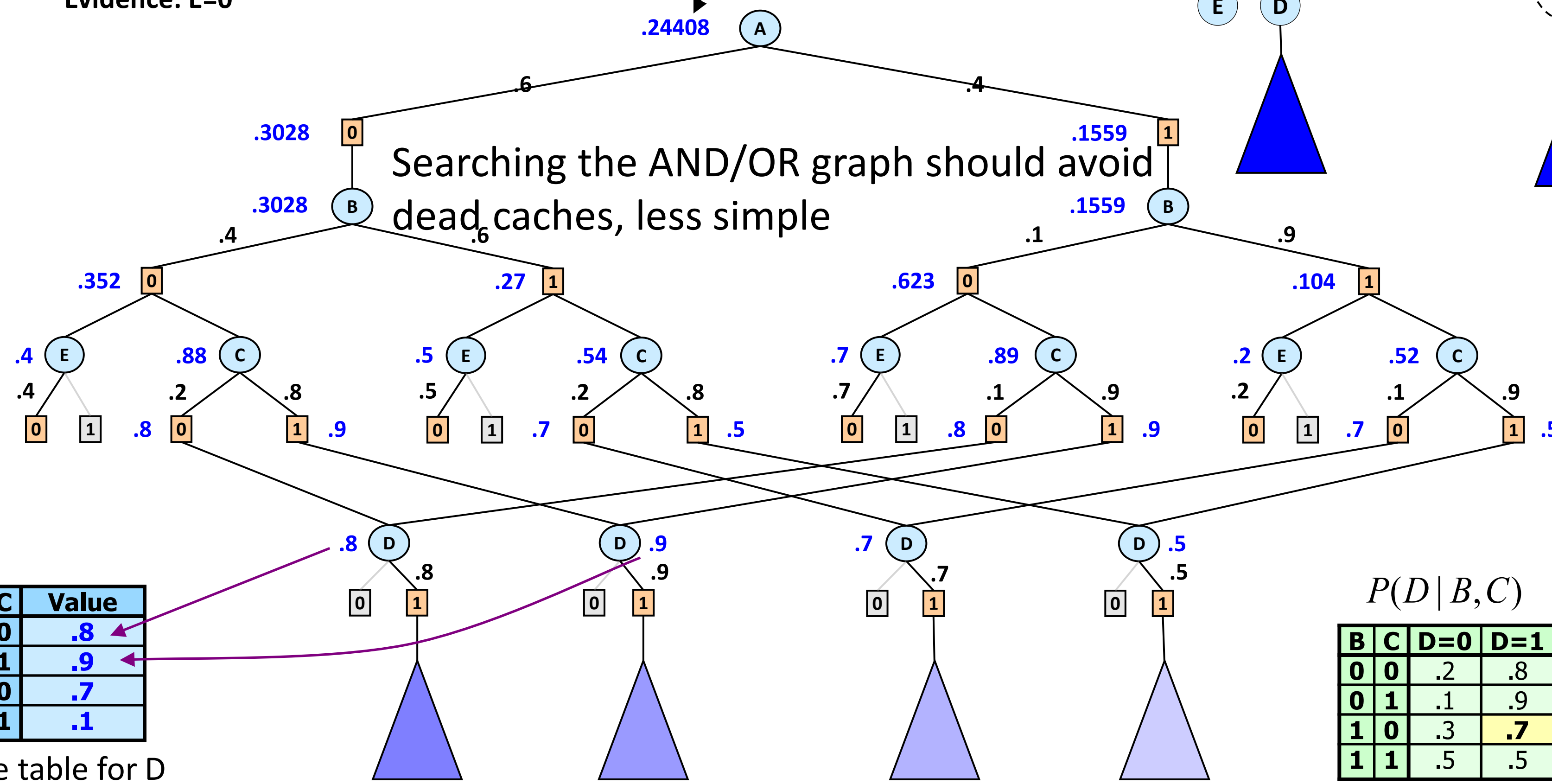
A	C=0	C=1
0	.2	.8
1	.7	.3

$P(A)$

A	P(A)
0	.6
1	.4



Result: P(D=1,E=0)



Searching the AND/OR graph should avoid dead caches, less simple

B	C	Value
0	0	.8
0	1	.9
1	0	.7
1	1	.1

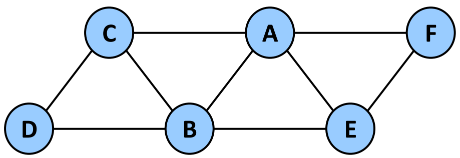
Cache table for D

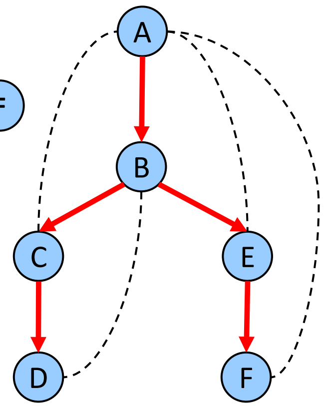
$P(D \mid B,C)$

B	C	D=0	D=1
0	0	.2	.8
0	1	.1	.9
1	0	.3	.7
1	1	.5	.5

Evidence: D=1

AND/OR Search Graph (Optimization)





A	B	f_1
0	0	2
0	1	0
1	0	1
1	1	4

A	C	f_2
0	0	3
0	1	0
1	0	0
1	1	1

A	E	f_3
0	0	0
0	1	3
1	0	2
1	1	0

A	F	f_4
0	0	2
0	1	0
1	0	0
1	1	2

B	C	f_5
0	0	0
0	1	1
1	0	2
1	1	4

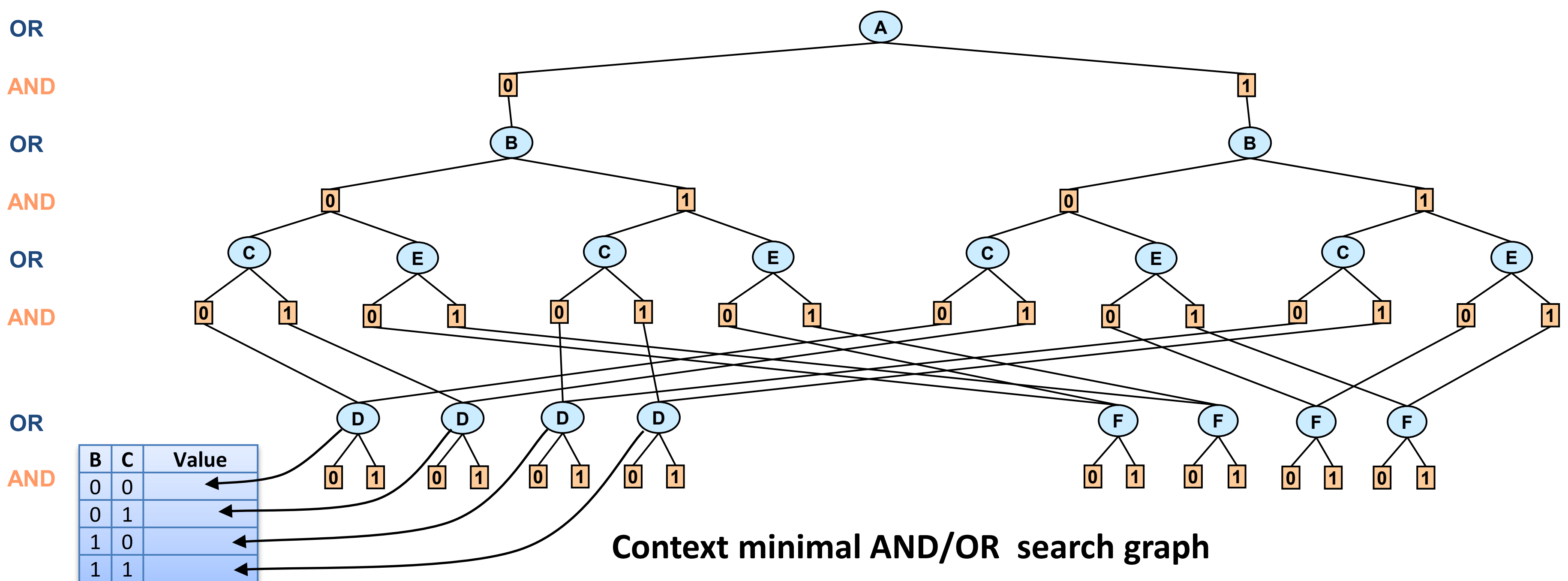
B	D	f_6
0	0	4
0	1	2
1	0	1
1	1	0

B	E	f_7
0	0	3
0	1	2
1	0	1
1	1	0

C	D	f_8
0	0	1
0	1	4
1	0	0
1	1	0

E	F	f_9
0	0	1
0	1	0
1	0	0
1	1	2

Objective function: $F^* = \min_x \sum_\alpha f_\alpha(x_\alpha)$



B	C	Value
0	0	
0	1	
1	0	
1	1	

Context minimal AND/OR search graph

Cache table for **D**

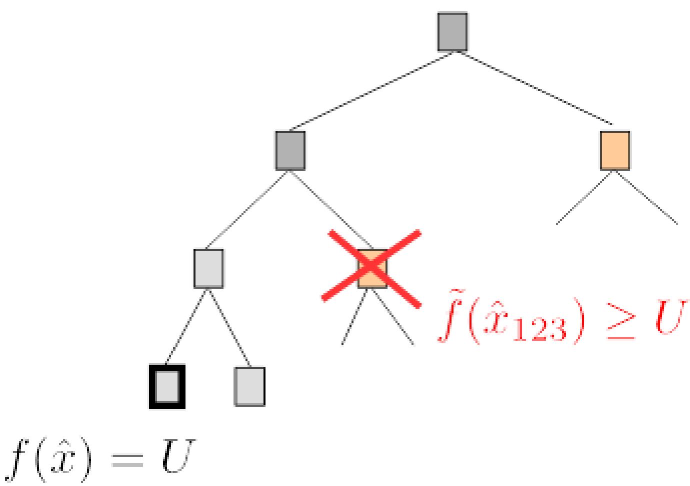
Basic Heuristic Search

We assume min-sum problems in the following

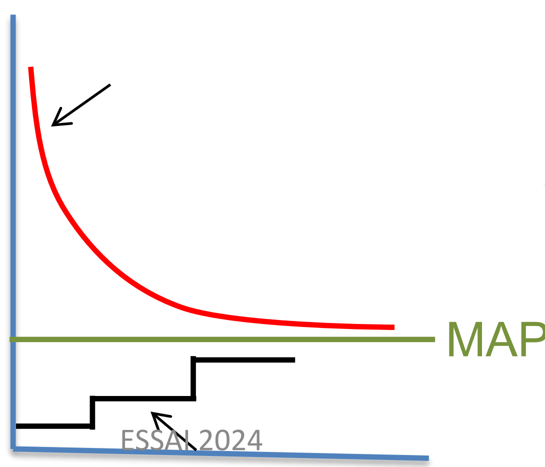
Heuristic function $\tilde{f}(\hat{x}_p)$ computes a lower bound on the best extension of partial configuration $\hat{x}_p$ and can be used to guide heuristic search.
We focus on:

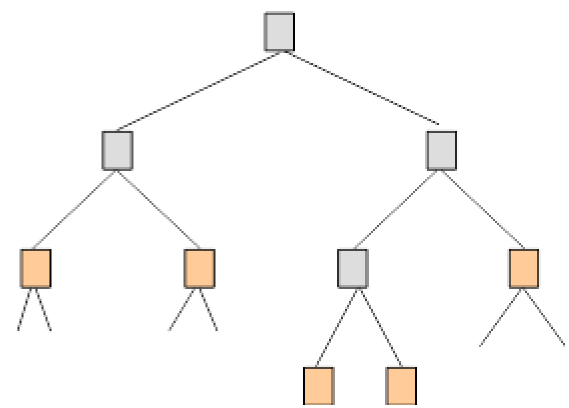
1. Branch-and-Bound
Use heuristic function $\tilde{f}(\hat{x}_p)$ to prune the depth-first search tree
Linear space

2. Best-First Search
Always expand the node with the lowest heuristic value $\tilde{f}(\hat{x}_p)$
Needs lots of memory



BnB is upper-bound anytime

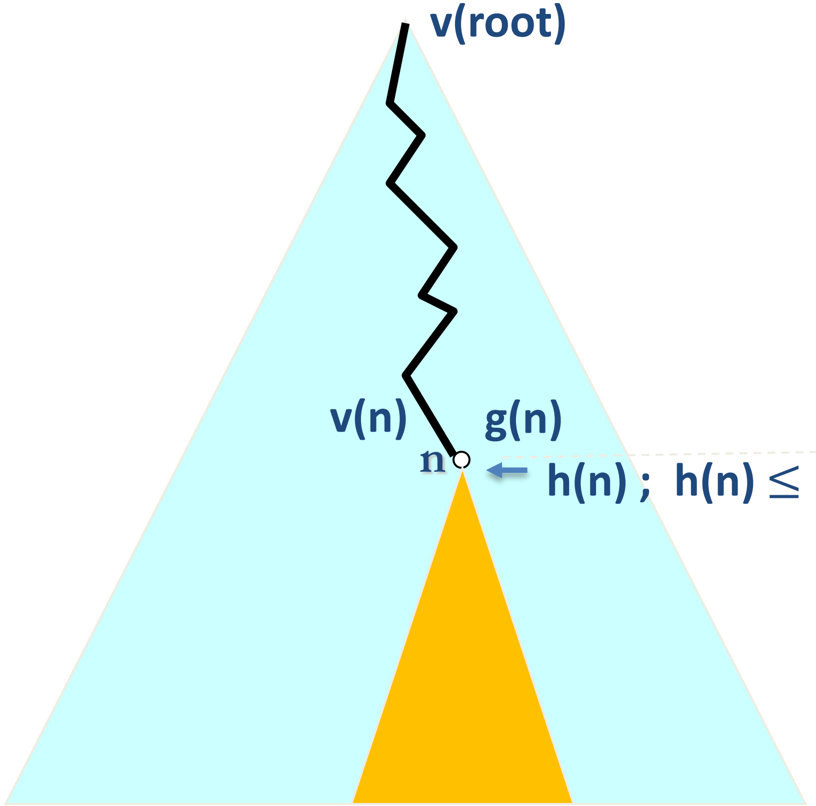




Basic Heuristic Search; **Best-First**

Task: compute v(root): MAP, Marginal, MMAP

Each node is a sub-problem
(defined by current conditioning)

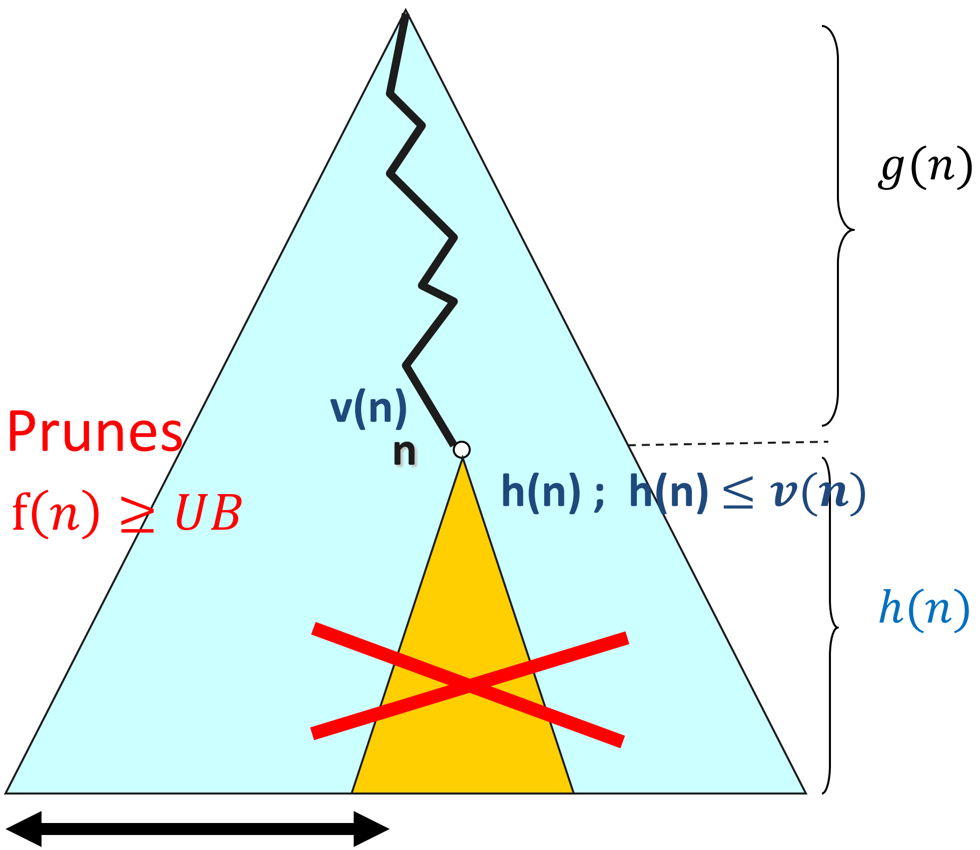


- **Best-First Algorithms, (A*)**
  - Expand nodes in OPEN list in order of min f(n)
  - Terminates with first full solution (for MAP)
- **Properties**
  - Optimal, if **h(n)** $\leq \boldsymbol{v}(\boldsymbol{n})$
  - Expands least set of nodes
  - exponential memory
  - **Not anytime solution for MAP**
  - **Yields lower bounds on value, anytime**

f(n) = g(n) + h(n) $\leq g(n) + v(n)$ =f*(n)

f(n) is a lower bound on best cost through n

Basic Heuristic Search; **Depth-First**



(UB) Upper Bound = best solution so far

- **Depth-First (B&B for MAP)**
  - Expand in dfs order
  - Update UB with each solution
  - Prunes if f(n) ≥ UB
- **Properties**
  - Can use only linear memory
  - Yields upper bounds anytime

Outline: Search

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AND/OR Heuristic Search

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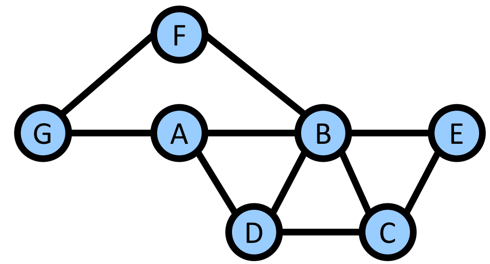
Search & Inference

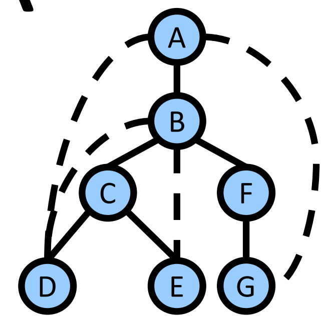
Value and Heuristic for AND/OR

Value $v(n)$: answer of the subtree rooted at node n

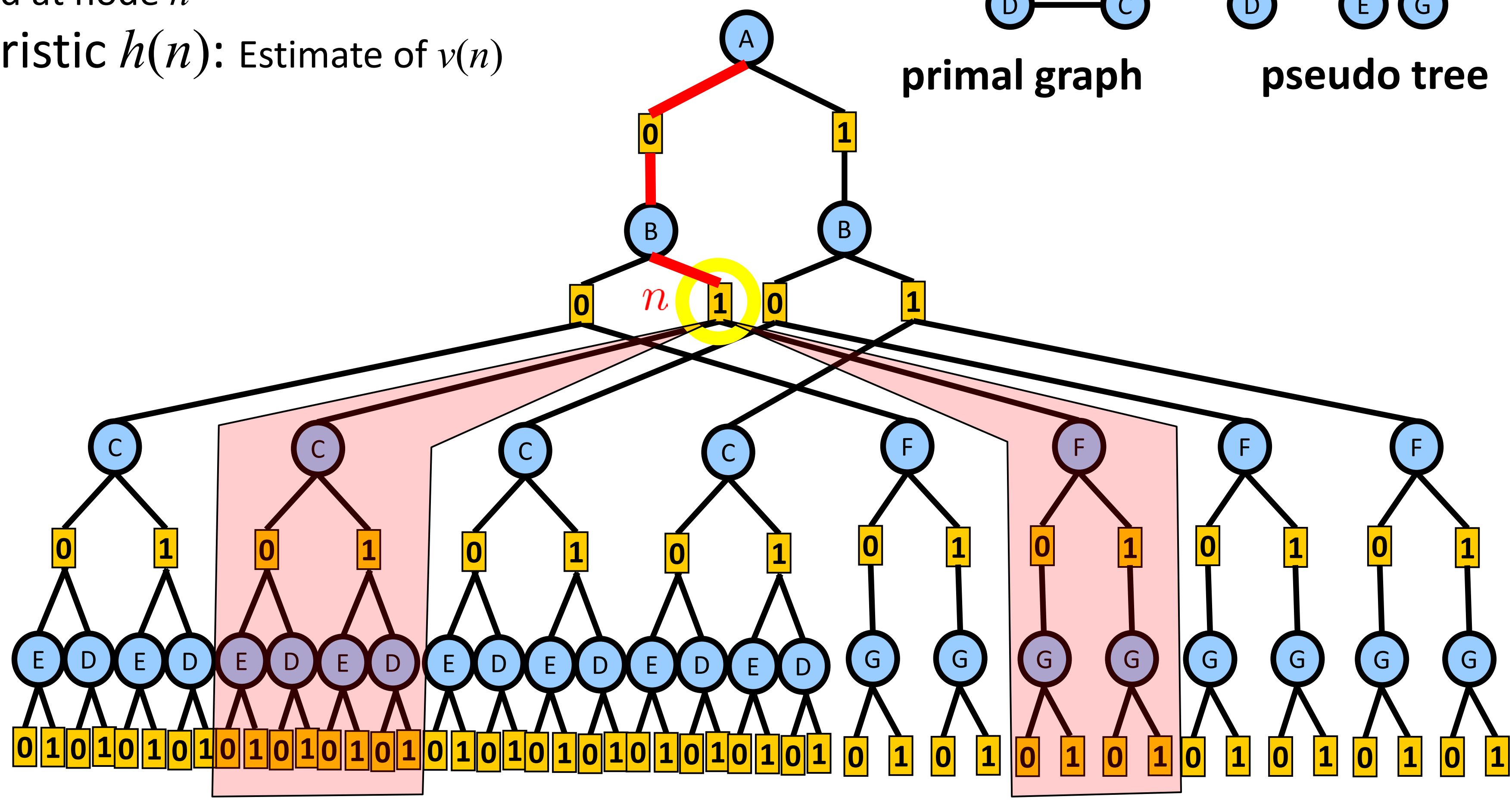
Heuristic $h(n)$: Estimate of $v(n)$



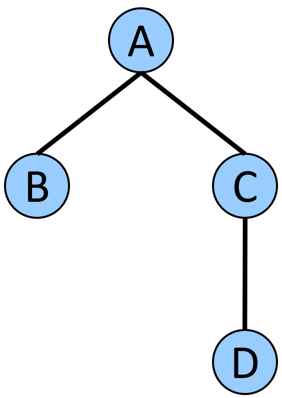
primal graph



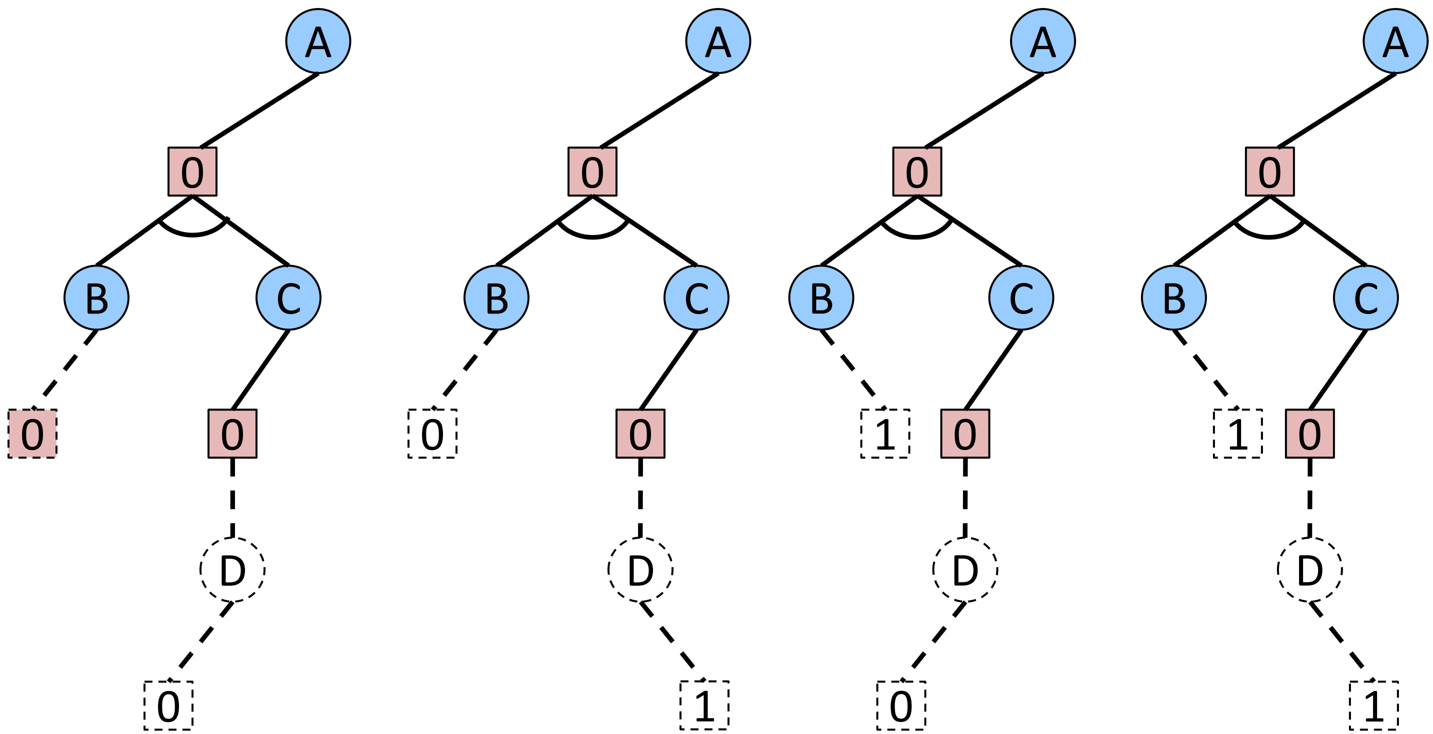
pseudo tree



Partial Solution Tree



Pseudo tree

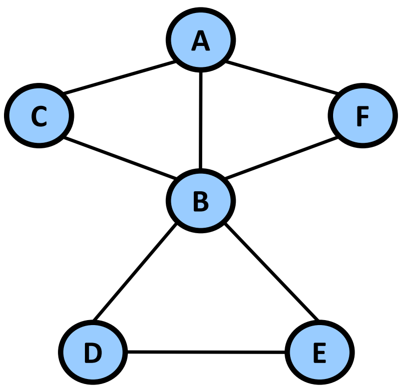


(A=0, B=0, C=0, D=0)   (A=0, B=0, C=0, D=1)   (A=0, B=1, C=0, D=0)   (A=0, B=1, C=0, D=1)

Extension(T’) – solution trees that extend T’
g(T’) = conditioned value of a node
V(T’) = the combined value below T’
f*(T’) = conditioned value through T’

Exact Evaluation Function

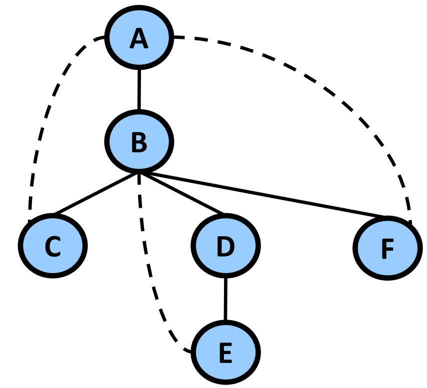
Conditioned value of a node

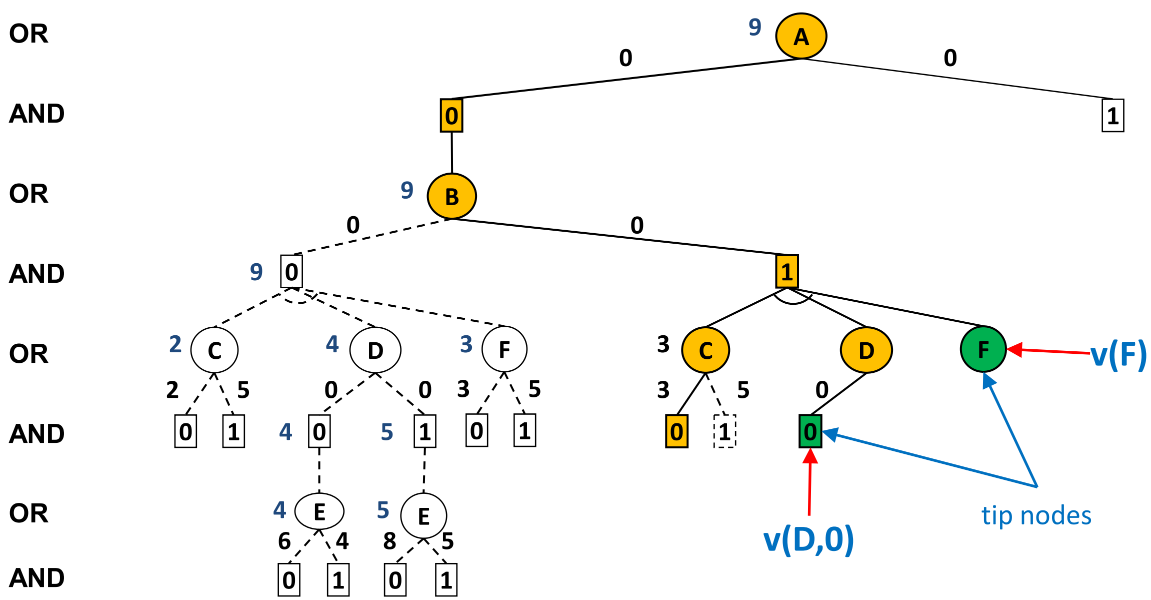


A	B	C	$f_1(ABC)$
0	0	0	2
0	0	1	5
0	1	0	3
0	1	1	5
1	0	0	9
1	0	1	3
1	1	0	7
1	1	1	2

A	B	F	$f_2(ABF)$
0	0	0	3
0	0	1	5
0	1	0	1
0	1	1	4
1	0	0	6
1	0	1	5
1	1	0	6
1	1	1	5

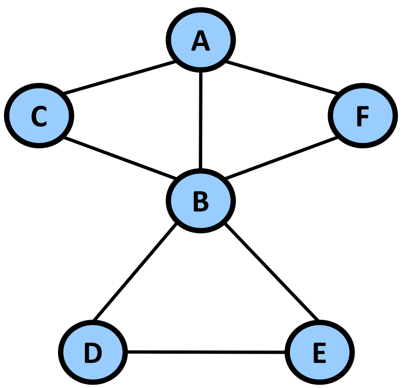
B	D	E	$f_3(BDE)$
0	0	0	6
0	0	1	4
0	1	0	8
0	1	1	5
1	0	0	9
1	0	1	3
1	1	0	7
1	1	1	4





$$f^*(T') = w(A,0) + w(B,1) + w(C,0) + w(D,0) + v(D,0) + v(F)$$

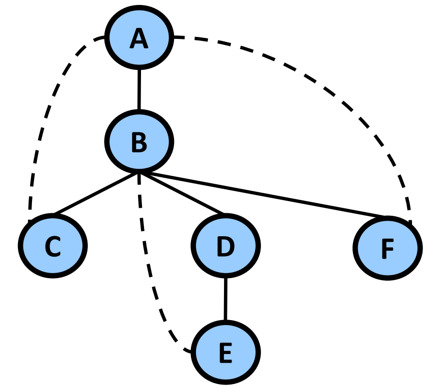
Heuristic Evaluation Function

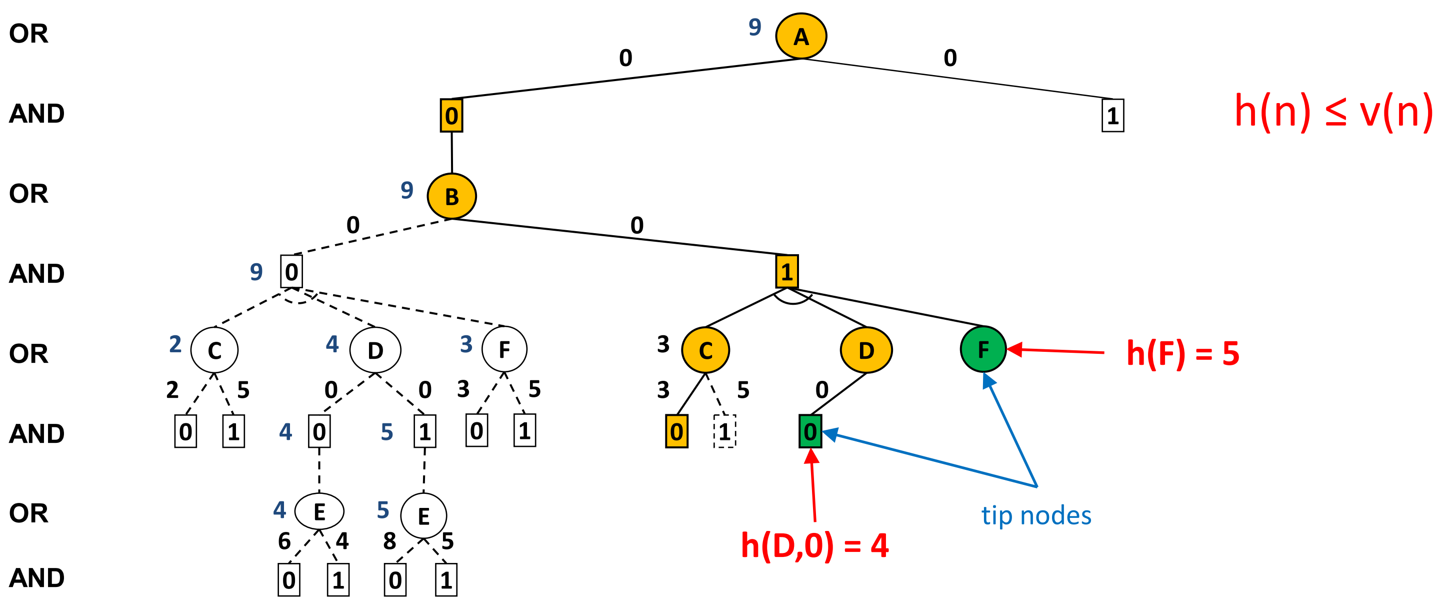


A	B	C	$f_1(ABC)$
0	0	0	2
0	0	1	5
0	1	0	3
0	1	1	5
1	0	0	9
1	0	1	3
1	1	0	7
1	1	1	2

A	B	F	$f_2(ABF)$
0	0	0	3
0	0	1	5
0	1	0	1
0	1	1	4
1	0	0	6
1	0	1	5
1	1	0	6
1	1	1	5

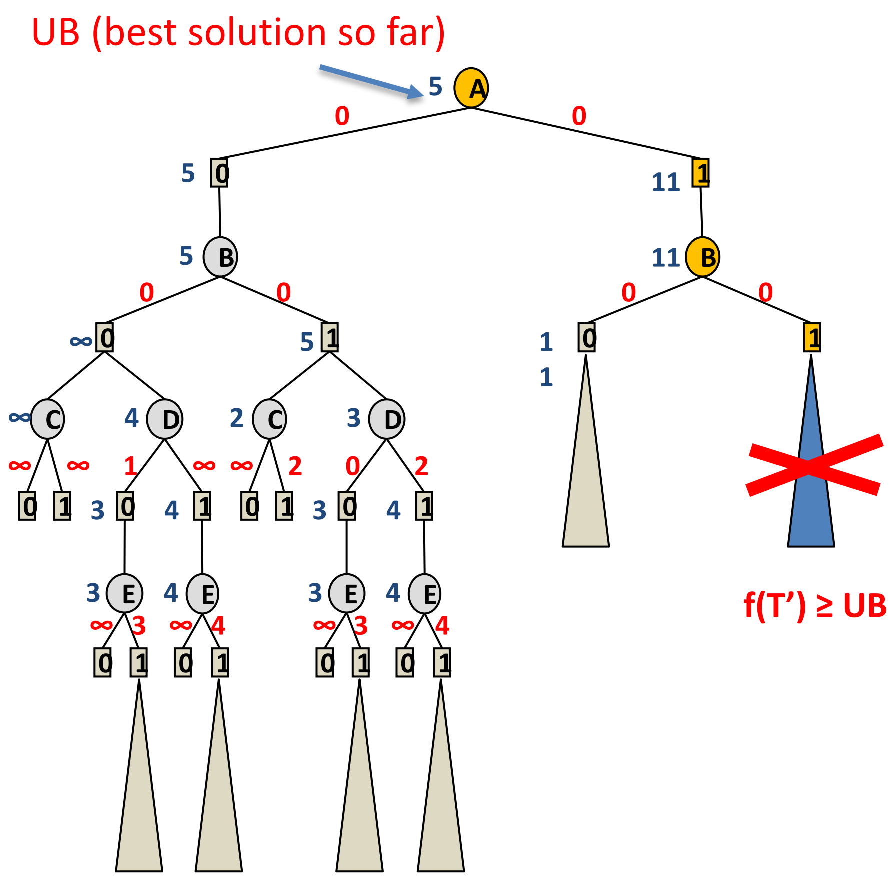
B	D	E	$f_3(BDE)$
0	0	0	6
0	0	1	4
0	1	0	8
0	1	1	5
1	0	0	9
1	0	1	3
1	1	0	7
1	1	1	4





$$f(T') = w(A,0) + w(B,1) + w(C,0) + w(D,0) + h(D,0) + h(F) = 12 \le f^*(T')$$

Depth-First AND/OR Branch-and-Bound



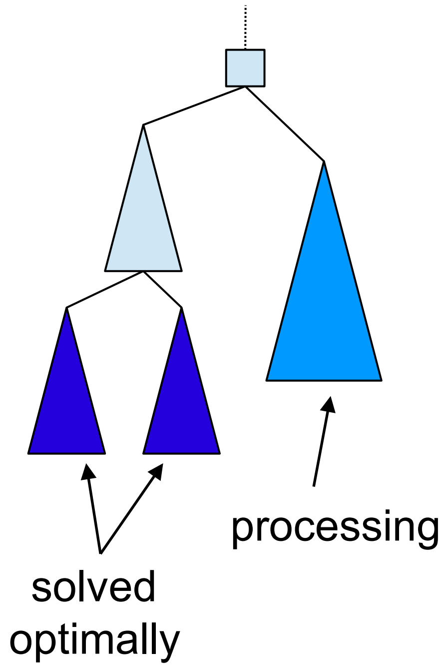
- Associate each node n with a heuristic lower bound h(n) on v(n)

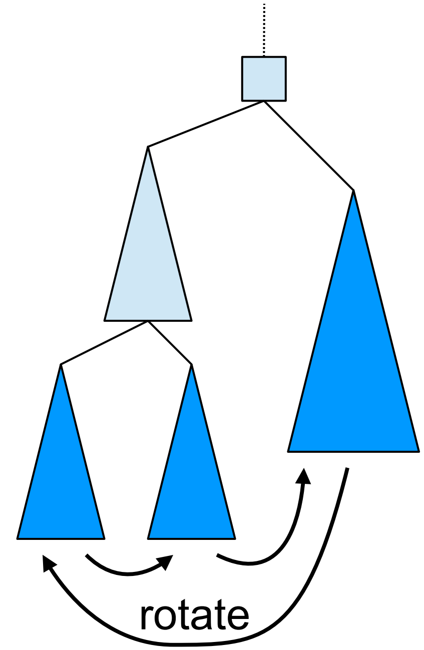
Algorithm AOBB:

- **EXPAND** (top-down)
  - Evaluate f(T') and prune search if f(T') ≥ UB
  - If not in cache, generate successors of the tip node n
- **PROPAGATE** (bottom-up)
  - Update value of the parent p of n
    - OR nodes: minimization
    - AND nodes: summation
  - Cache value of n based on context

Anytime Performance

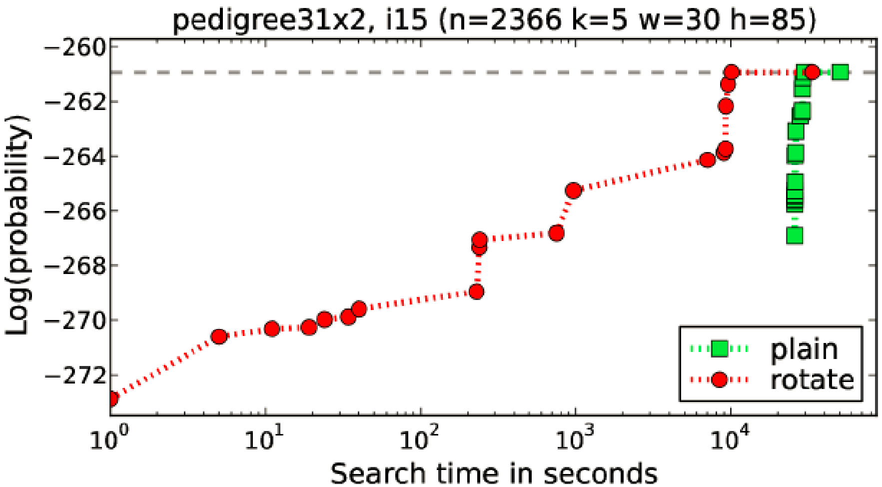
- OR Branch-and-Bound is anytime
- But AND/OR breaks anytime behavior of depth-first scheme:
  - First anytime solution delayed until last sub-problem starts processing
- **Breadth-Rotating AOBB:**
  - Take turns processing sub-problems
    - Limit number of expansions per visit
  - Solve each sub-problem depth-first
    - Maintain favorable complexity bounds

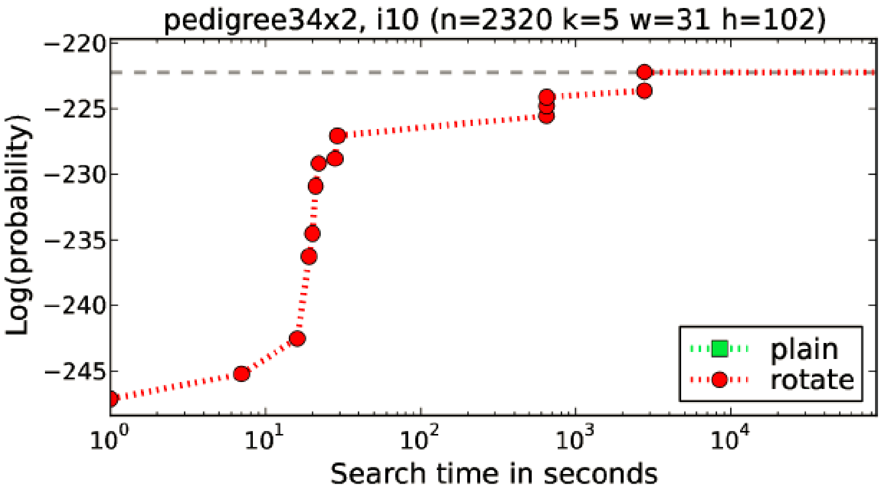




[Otten and Dechter, 2012]

Anytime Performance





- **Breadth-Rotating AOBB:**
  - Take turns processing sub-problems
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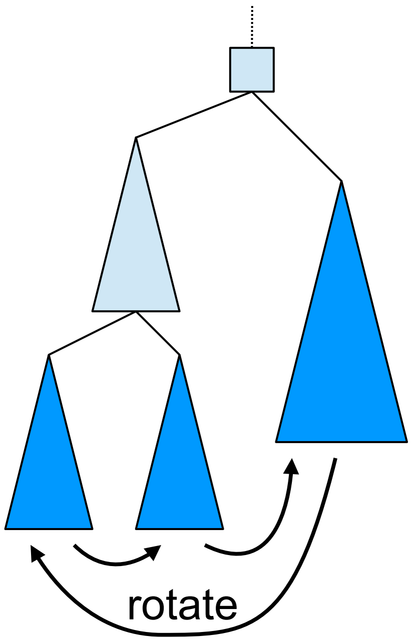
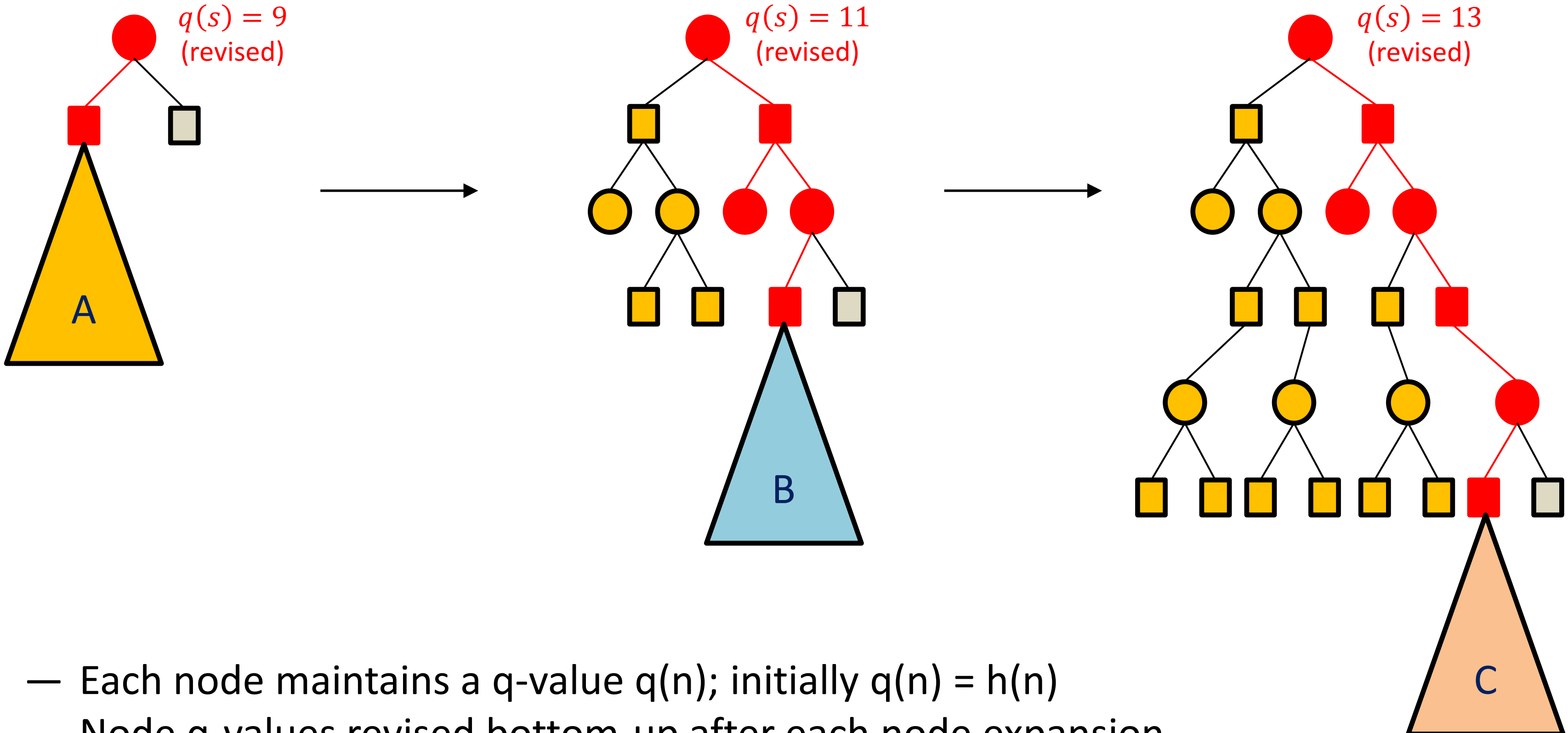


[Otten and Dechter, 2012]

AOBF: Best-First AND/OR Search



- Each node maintains a q-value q(n); initially q(n) = h(n)
- Node q-values revised bottom-up after each node expansion
- Update current best partial solution subtree (a tip node expanded next)
- All expanded nodes are stored in memory
- Search terminates with optimal solution (cost)

[Marinescu and Dechter, 2006; 2009]

AOBF: Best-First AND/OR Search

- AO*-traverses the context-minimal AND/OR graph
  - All nodes expanded are stored in memory
  - Each node maintains a q-value: q(n), (Best lower bound below n)

- Node q-values are revised bottom-up after each expansion
  - OR: minimization: $q(n) = \min_{n' \in succ(n)} \left(w(n, n') + q(n') \right)$
  - AND: summation: $q(n) = \Sigma_{n' \in succ(n)} q(n')$, (initially, q(n) = h(n))

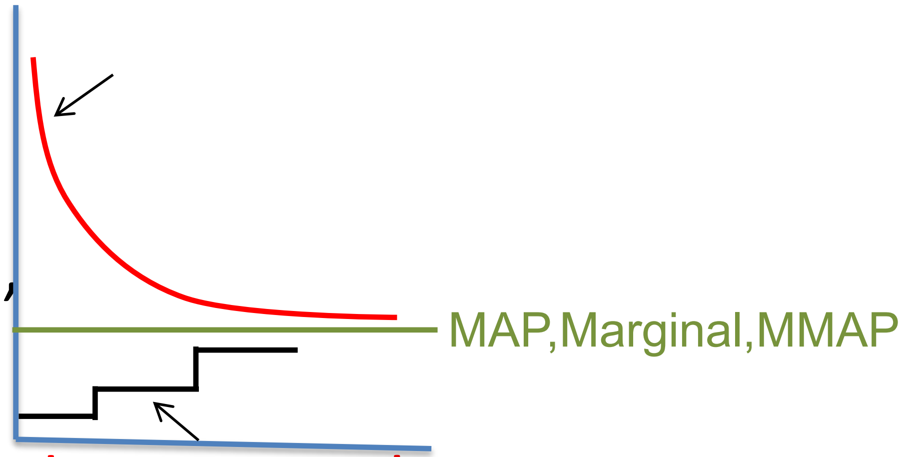
[Marinescu and Dechter, 2006; 2009]

AOBF versus AOBB

- **AOBF** expands a smallest subset of the AO search space
  - This translates into significant time savings

- **AOBB** can use far less memory by avoiding dead-caches, whereas **AOBF** keeps in memory the explicated search graph

- **AOBB** (**BRAOBB**) is anytime,



- **AOBF generates lower bounds anytime, but not anytime solutions (configuration)**

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AND/OR Heuristic Search

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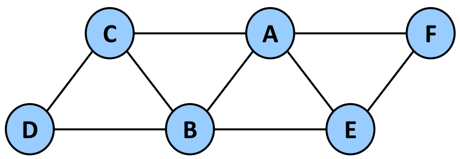
Search & Inference

Heuristics for Graphical Models



Given a cost function:

$$f(a, \dots, e) = f(a) + f(a, b) + f(a, c) + f(a, d) + f(b, c) + f(b, d) + f(b, e) + f(c, e)$$

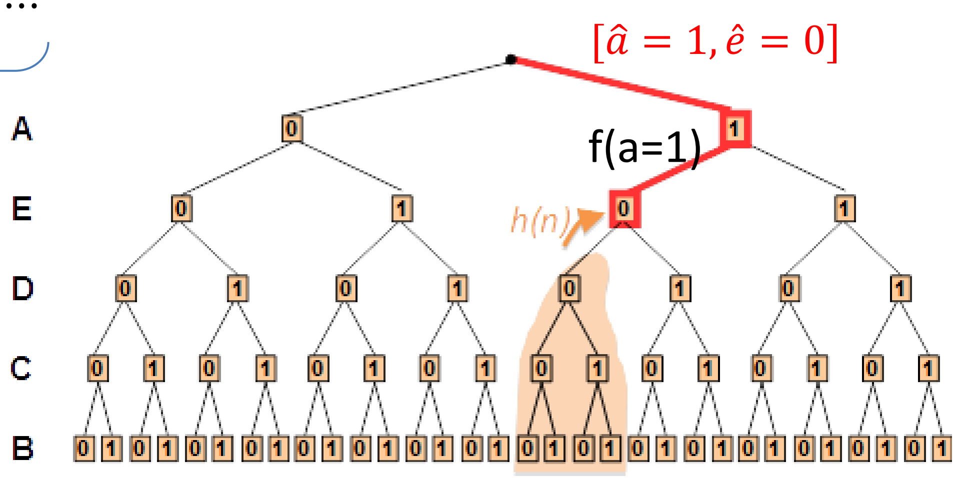
define an evaluation function over a partial assignment as the cost of its best extension:

$$f^*(\hat{a}, \hat{e}, D) = \min_{b,c} F(\hat{a}, b, c, D, \hat{e})$$

$$= f(\hat{a}) + \min_{b,c} f(\hat{a}, b) + f(\hat{a}, c) + \cdots$$

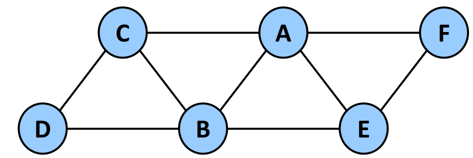
(h*=v)

$$= g(\hat{a}, \hat{e}, D) + h^*(\hat{a}, \hat{e}, D)$$

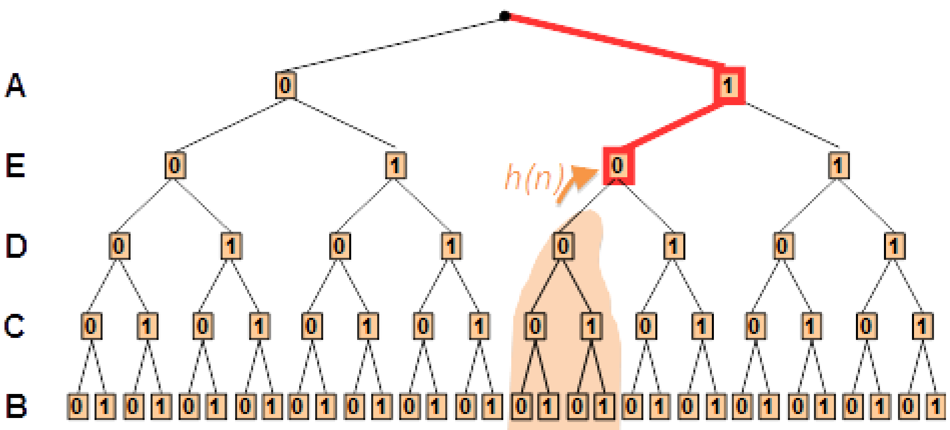


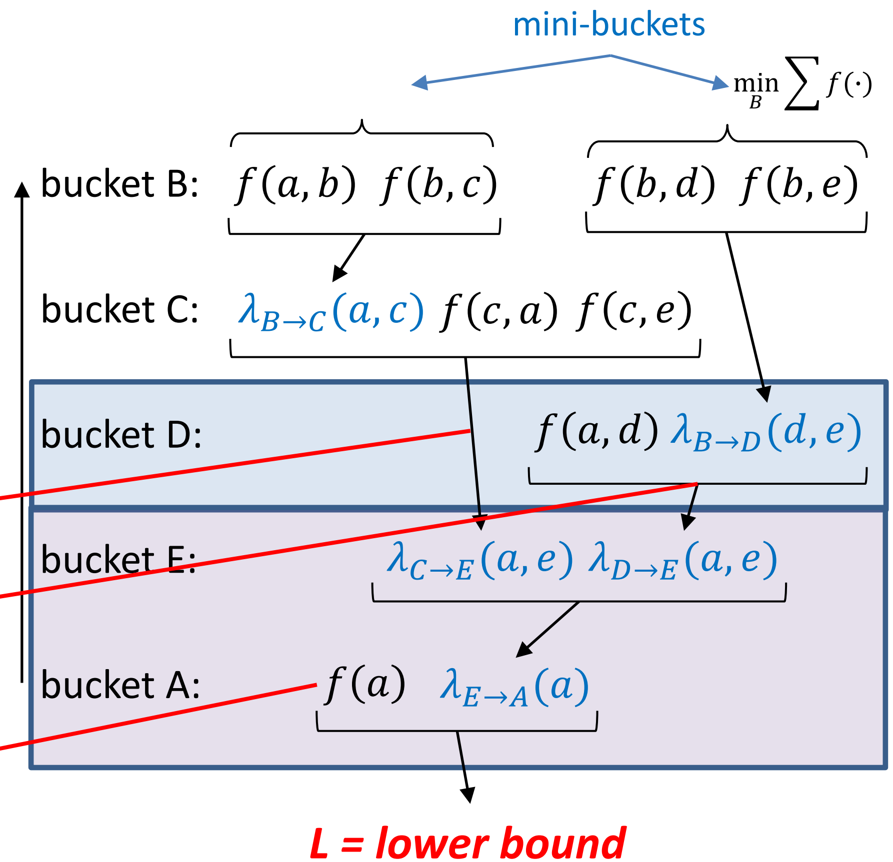
[Kask and Dechter, 2001]

Static Mini-Bucket Heuristics



Given a partial assignment, $[\hat{a} = 1, \hat{e} = 0]$
(weighted) mini-bucket gives an admissible heuristic:





cost to go:

$$h(\hat{a}, \hat{e}, D) = \lambda_{C \to E}(\hat{a}, \hat{e}) + f(\hat{a}, D) + \lambda_{B \to D}(D, \hat{e})$$

(admissible: $h(\hat{a}, \hat{e}, D) \leq h^*(\hat{a}, \hat{e}, D)$)

cost so far:

$$g(\hat{a}, \hat{e}, D) = f(A = \hat{a})$$

Properties of the MBE Heuristics

- MBE heuristic is monotone, admissible
- Computed in linear time (during search)
- Important:
  - Heuristic strength can vary by MBE(i)
  - Higher i-bound → more pre-processing → more accurate heuristic → less search
- Allows controlled trade-off between pre-processing and search
- Can be computed statically or dynamically during search

Review: Weighted Mini-bucket

[Liu & Ihler 2011]

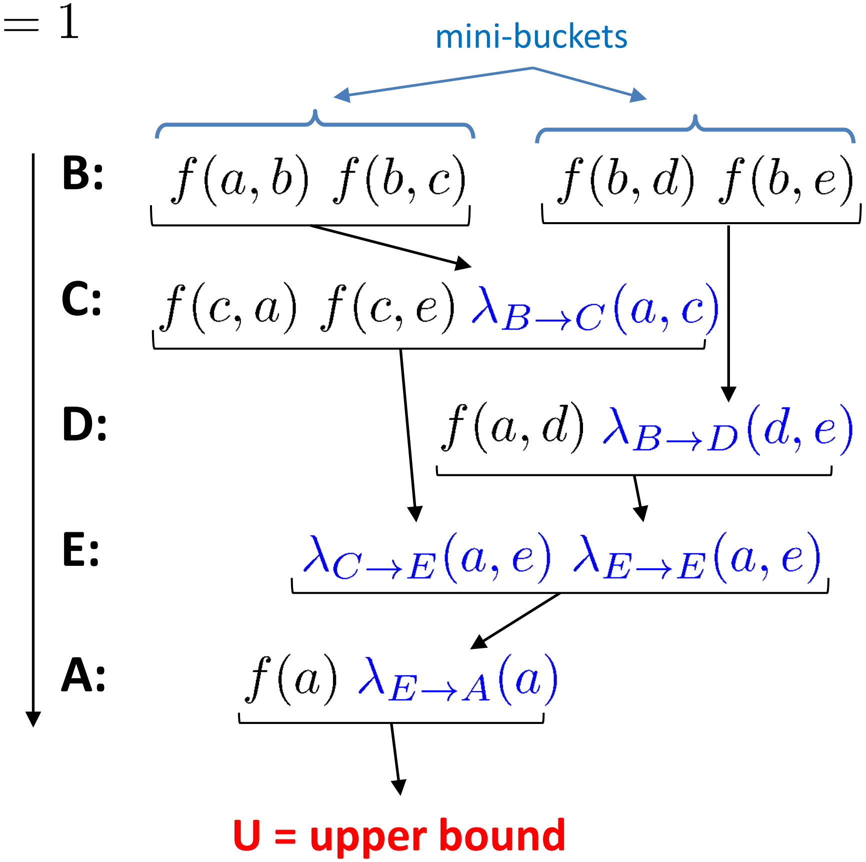
For Sum-Inference

$$\lambda_{B \to C} = \sum_b^{w_{B1}} f(a,b) \cdot f(b,c) \qquad w_{B1} + w_{B2} = 1$$

$$\lambda_{B \to D} = \sum_b^{w_{B2}} f(b,d) \cdot f(b,e)$$

$$\lambda_{C \to E} = \sum_c f(c,a) \cdot f(c,e) \cdot \lambda_{B \to C}$$

⋮



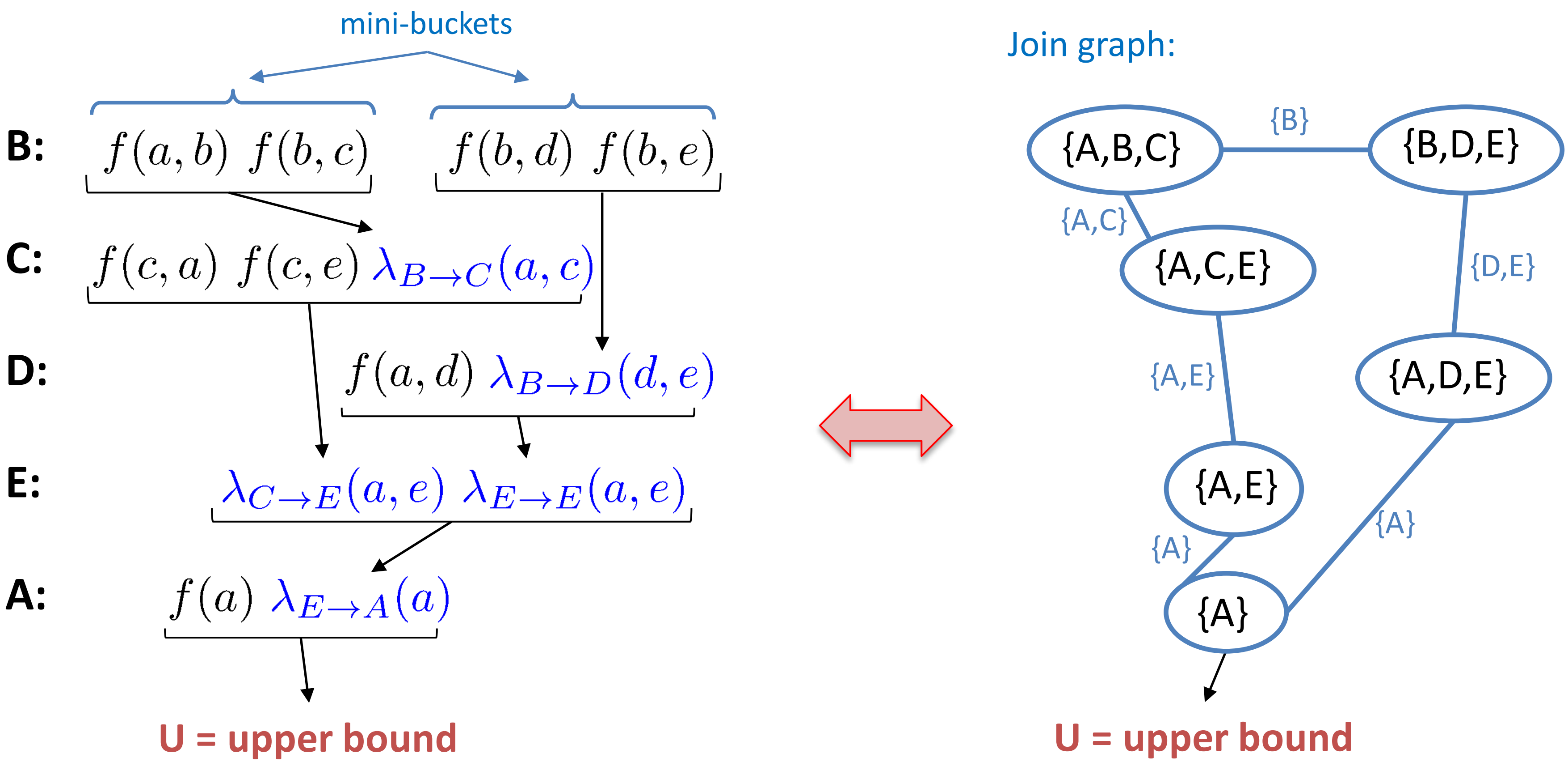
Compute downward messages
using weighted sum

Upper bound if all weights positive
(corresponding lower bound if only one positive, rest negative)

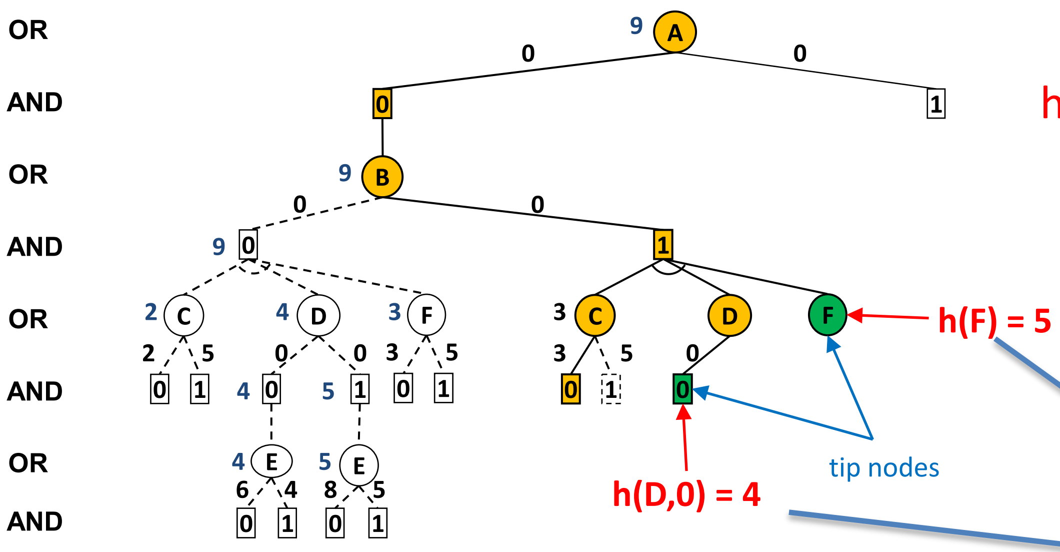
Review:MBE+Moment Matching

For all queries

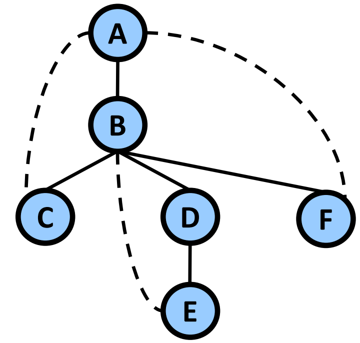
- Mini-bucket elimination defines regions with bounded complexity



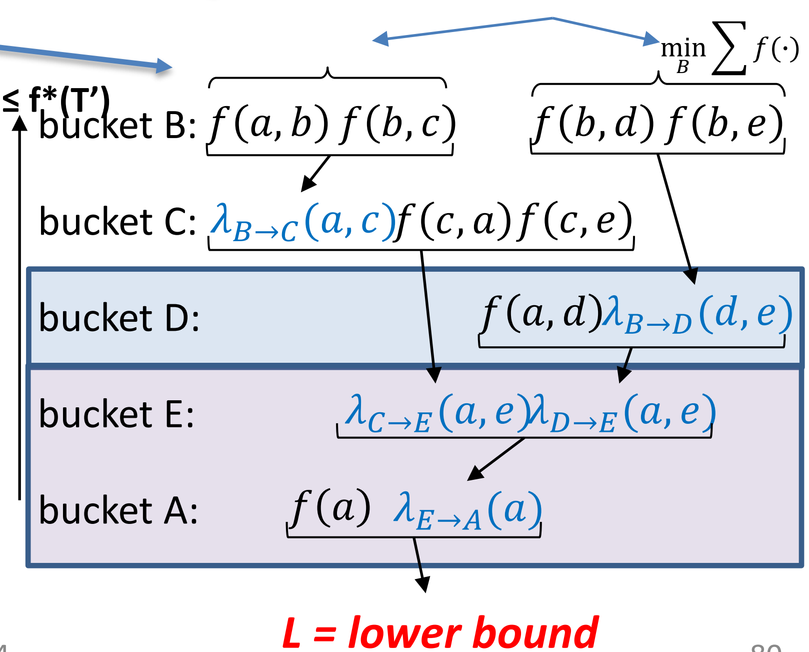
MBE Heuristic Guides AO Search



$h(n) \le v(n)$



f(T') = w(A,0) + w(B,1) + w(C,0) + w(D,0) + h(D,0) + h(F) = 12 ≤ f*(T')



Outline: Search

- AND/OR Search Trees
- AND/OR Search Graphs
- Pseudo trees generation

AND/OR Search spaces

- Basic Search (depth and Best)
- AND/OR Depth and Best Heuristic Search
- The Guiding MBE Heuristic
- **Searching for Mixed tasks (MMAP, ID)**

AND/OR Heuristic Search

- Hybrid of Search and Inference

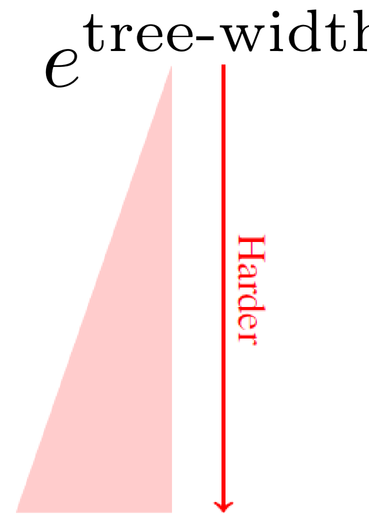
Search & Inference

Probabilistic Reasoning Problems

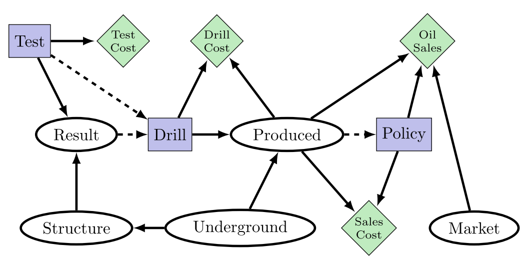
- Exact Inference by elimination or search
- Complexity:

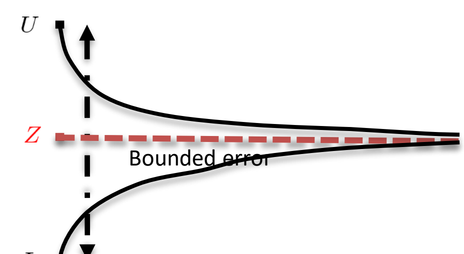
Causal effects

Max-Inference:	$f(x^*) = \max_x \prod_\alpha f_\alpha(x_\alpha)$
Sum-Inference:	$Z = \sum_x \prod_\alpha f_\alpha(x_\alpha)$
Mixed-Inference (MMAP):	$f_M(x_M^*) = \max_{x_M} \sum_{x_S} \prod_\alpha f_\alpha(x_\alpha)$
Mixed-Inference (MEU):	$\text{MEU} = \max_{D_1,\ldots,D_m} \sum_{X_1,\ldots X_n} (\prod_{P_i \in P} P_i) \times (\sum_{r_i \in R} r_i)$

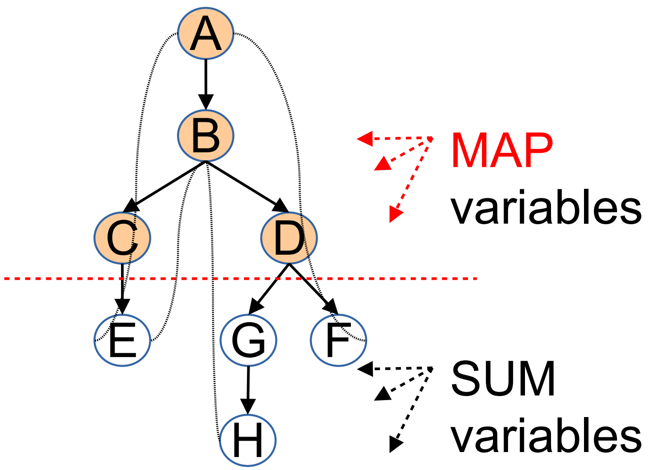


Influence diagrams & planning

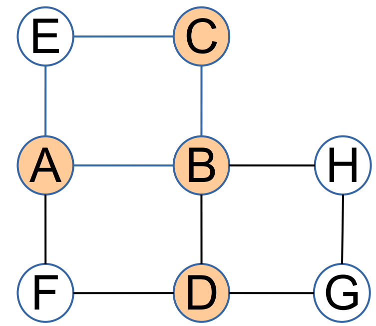




AND/OR Search for Marginal MAP

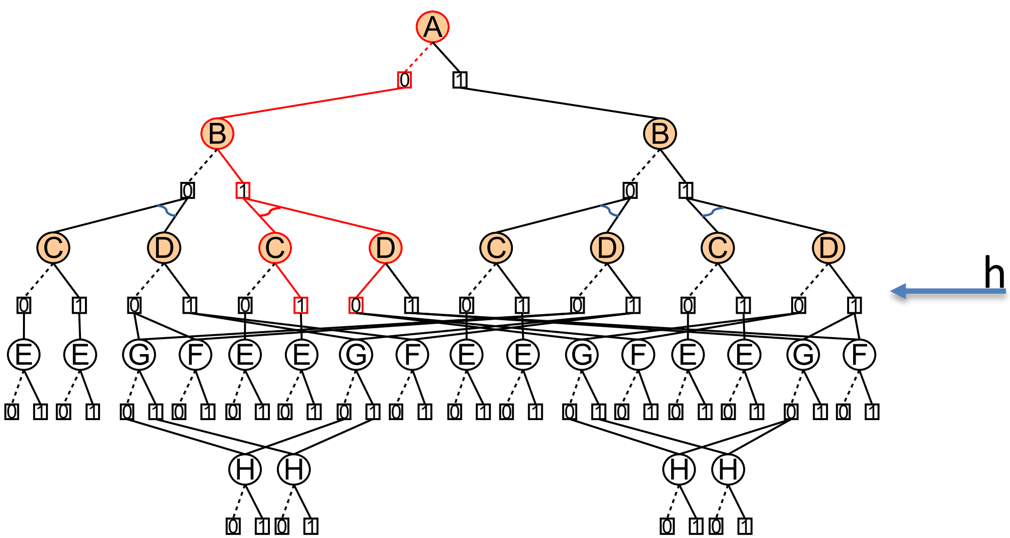


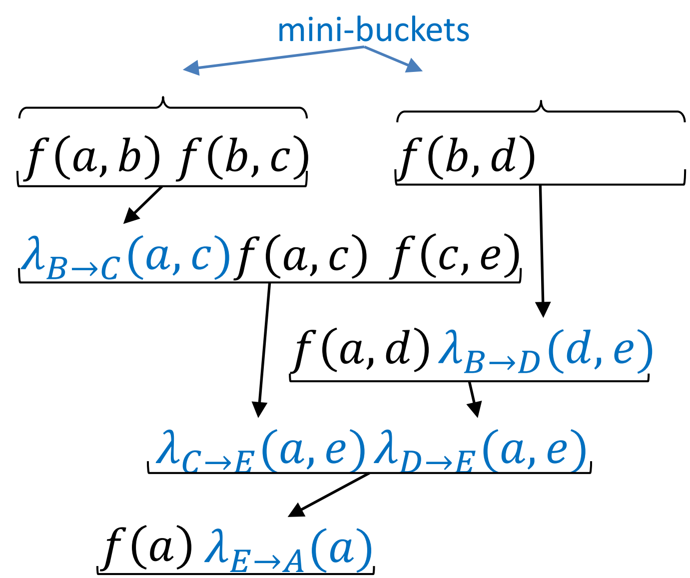
constrained pseudo tree



primal graph

Anytime Depth+Best to yield upper and lower bounds

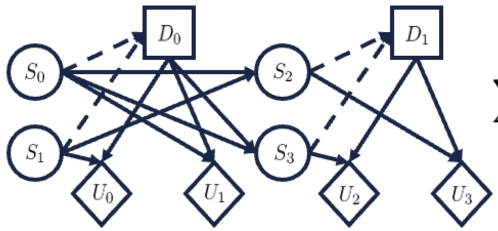




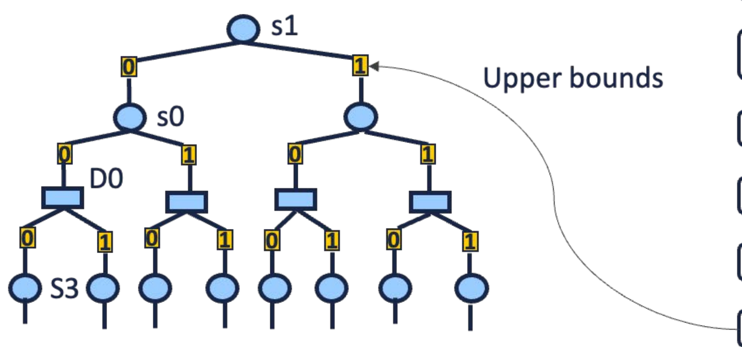
[Marinescu, Dechter and Ihler, 2014]

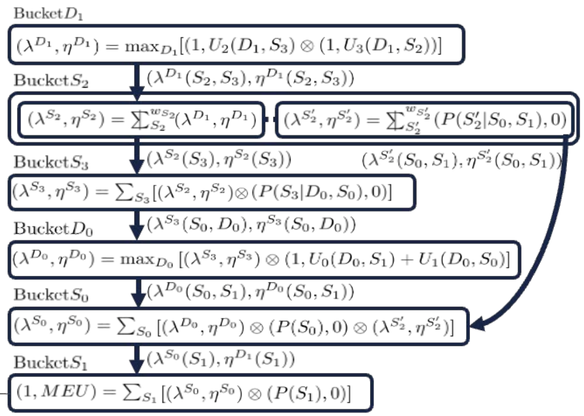
AND/OR Search for Influence Diagrams

Heuristic AND/OR search with decomposition bounds



$$\sum_{S_0,S_1} \max_{D_0} \sum_{S_2,S_3} \max_{D_1} [\prod_{P_i \in \mathbf{P}} P_i][\sum_{U_i \in \mathbf{U}} U_i][\prod_{\Delta_i \in \mathbf{\Delta}} \Delta_i]$$





[Marinescu 2010]

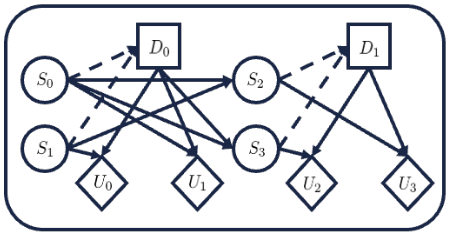
AND/OR Search Graph for Influence Diagrams

[Lee et.al 2019]

Weighted Mini-bucket elimination bound for Influence Diagrams

AND/OR Search for Influence Diagrams

Influence Diagram [Howard and Matheson 1981]



[Jensen et al 1994; Maua et. al 2012]

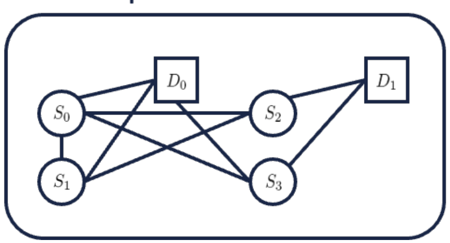
Valuation Algebra for Influence Diagrams

$$\mathbf{\Psi} := \{(P_i, 0) | P_i \in \mathbf{P}\} \cup \{(1, U_i) | U_i \in \mathbf{U}\}$$

$$\Psi_1 \otimes \Psi_2 := (P_1 P_2, P_1 V_2 + P_2 V_1)$$

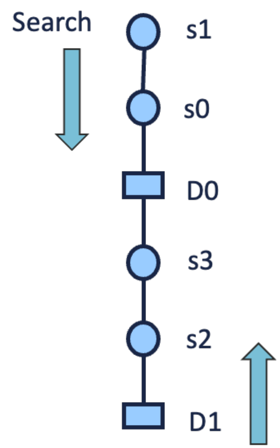
$$\textstyle\sum_{\mathbf{Y}}^{\mathbf{w}} \Psi := (\sum_{\mathbf{Y}}^{\mathbf{w}} P, \sum_{\mathbf{Y}}^{\mathbf{w}} V)$$

Primal Graph



[Freuder and Quinn 1985]

Pseudo-tree

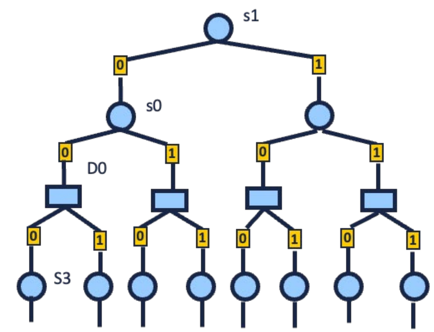


Inference [Dechter 1999]

By bucket elimination

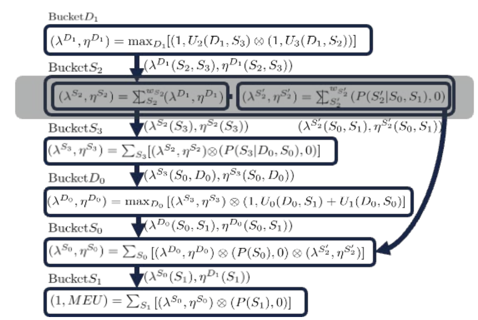
[Dechter et al 2007; Marinescu, et al 2010]

AND/OR Search Space



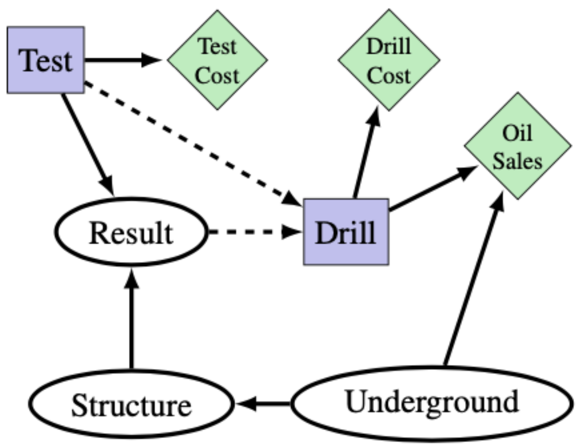
[Dechter et al 2003; Liu et al 2011; Lee et al 2019]

Decomposition bounds from bucket tree



AND/OR Search for Influence Diagrams

- Use perfect recall order to enumerate search tree



$p(U)$	$U=0$	$U=1$	$U=2$
	0.5	0.3	0.2

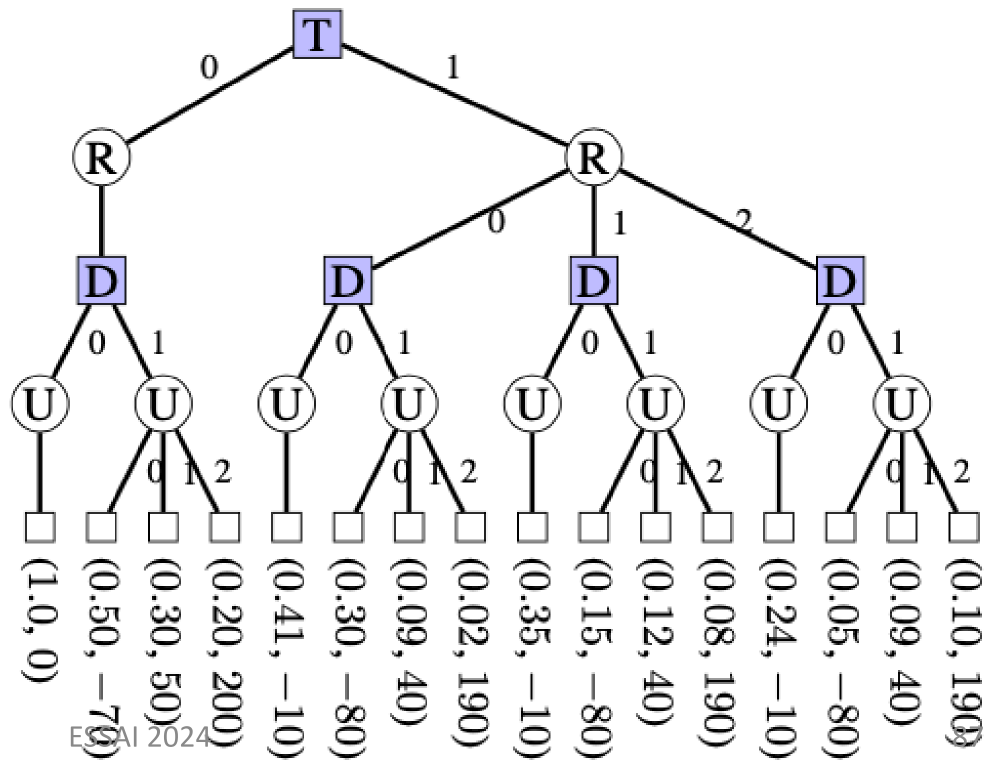
$p(S\|U)$	$S=0$	$S=1$	$S=2$
$U=0$	0.6	0.3	0.1
$U=1$	0.3	0.4	0.3
$U=2$	0.1	0.4	0.5

$p(R|S,T) = \mathbb{1}[R = ST]$

$u_1(T)$	$T=0$	$T=1$
	\$0	$-$\$10k

$u_2(D)$	$D=0$	$D=1$
	\$0	$-$\$70k

$u_3(D,U)$	
$D=0$	\$0
$DU=10$	\$0
$DU=11$	\$120k
$DU=12$	\$270k



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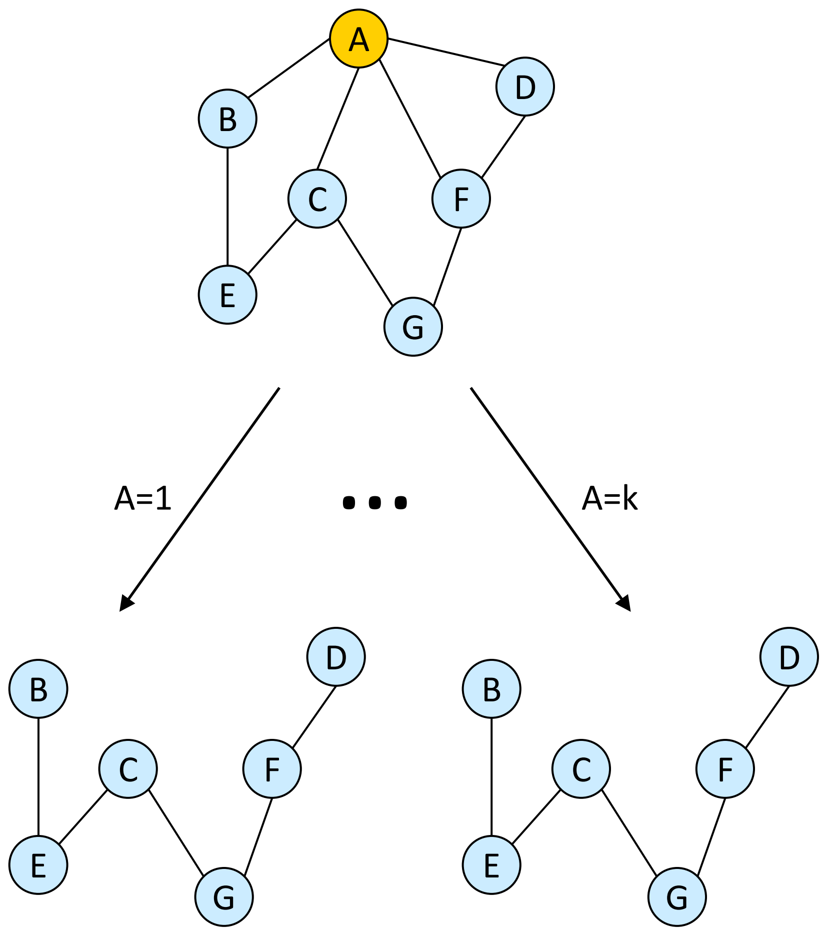
Search & Inference

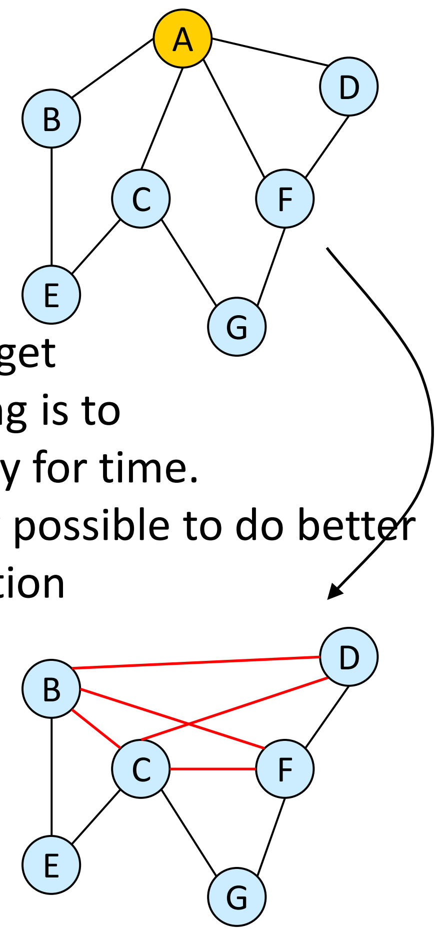
Conditioning versus Elimination

Conditioning (search)



k “sparser” problems

Elimination (inference)

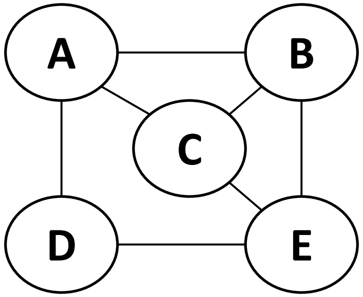


The main target
In conditioning is to
Trade memory for time.
It is not really possible to do better
Then elimination

1 “denser” problem

Hybrid: Cutset-Conditioning

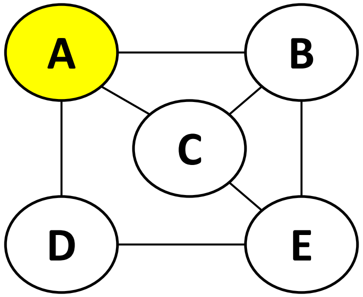
Variable Branching by Conditioning



Hybrid: Cutset-Conditioning

Variable Branching by Conditioning

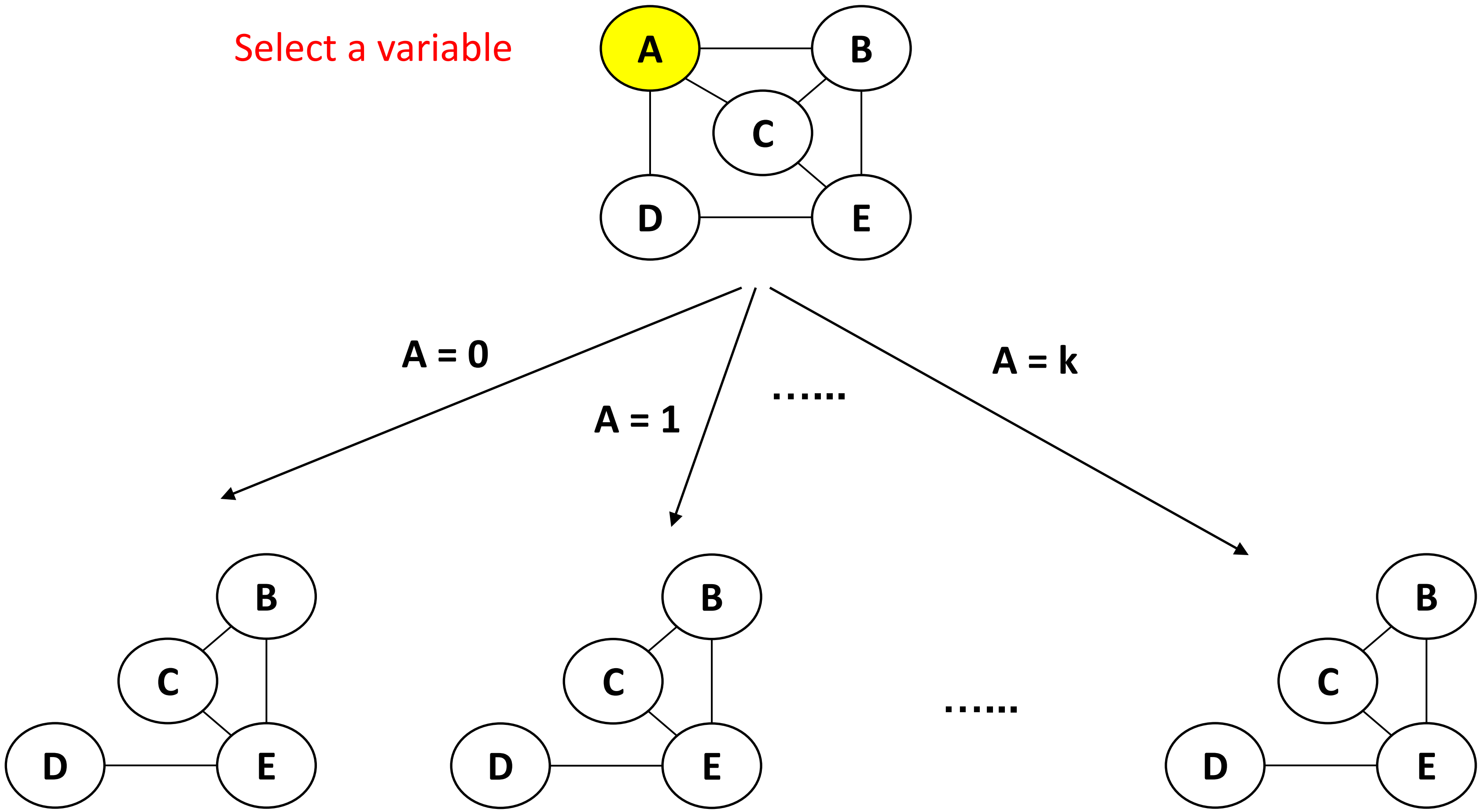
Select a variable



Hybrid: Cutset-Conditioning

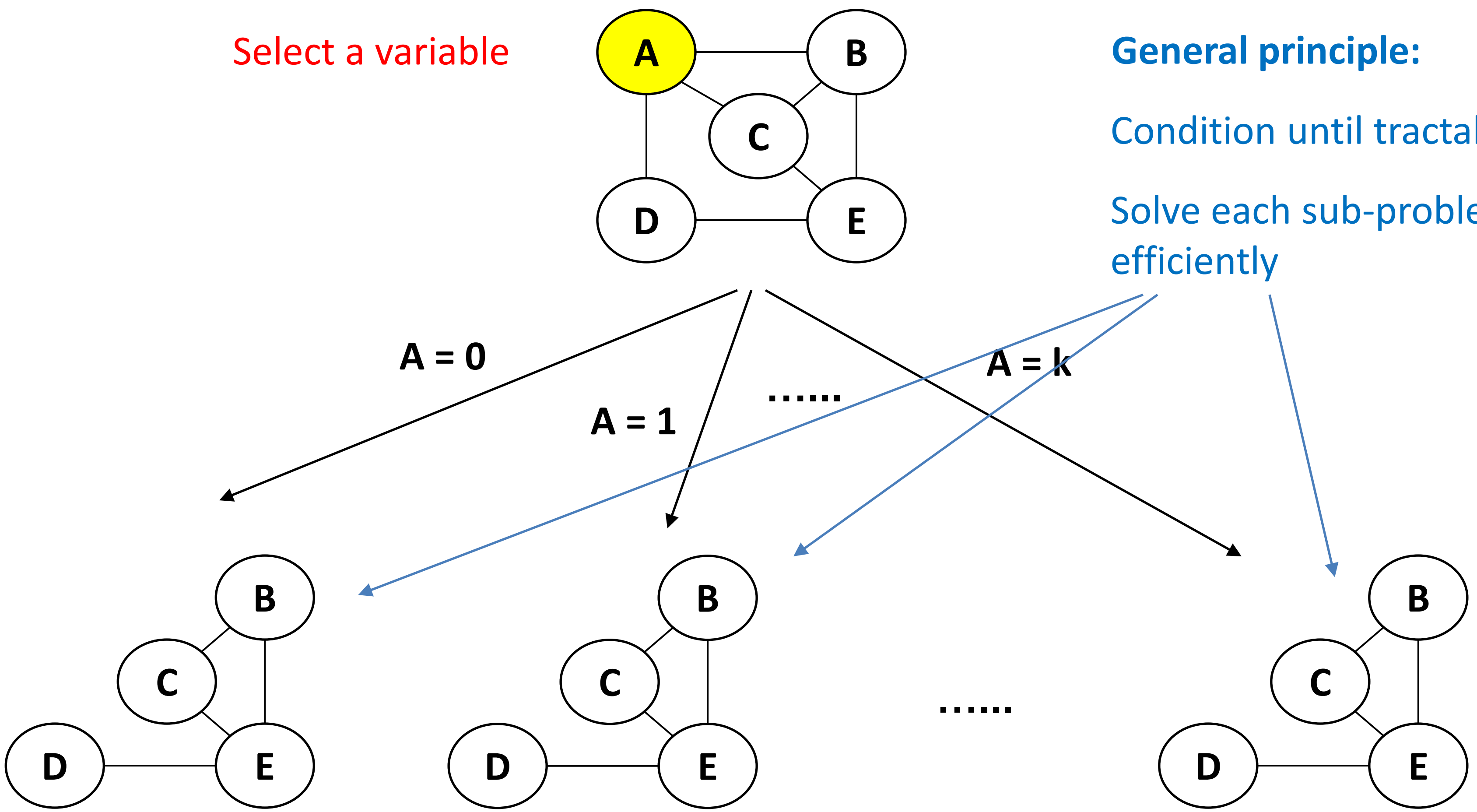
Variable Branching by Conditioning

Select a variable



Hybrid: Cutset-Conditioning

Variable Branching by Conditioning



General principle:

Condition until tractable

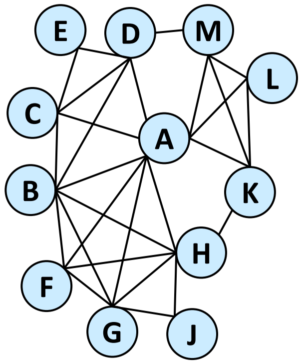
Solve each sub-problem efficiently

Hybrids Variants

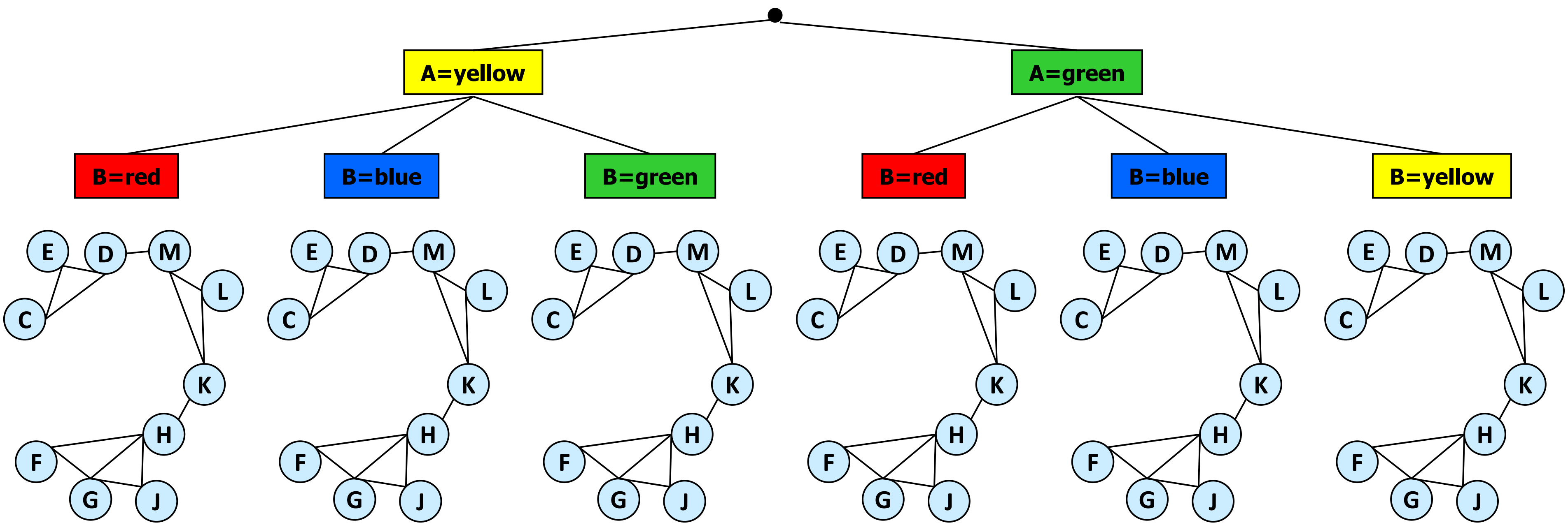
- **Condition, condition, condition,** … and then only eliminate (w-cutset, cycle-cutset VEC(i))
- **Eliminate, eliminate, eliminate,** … and then only search
- **Alternate** conditioning and elimination steps (elim-cond(i), ALT-VEC(i))

OR w-Cutset

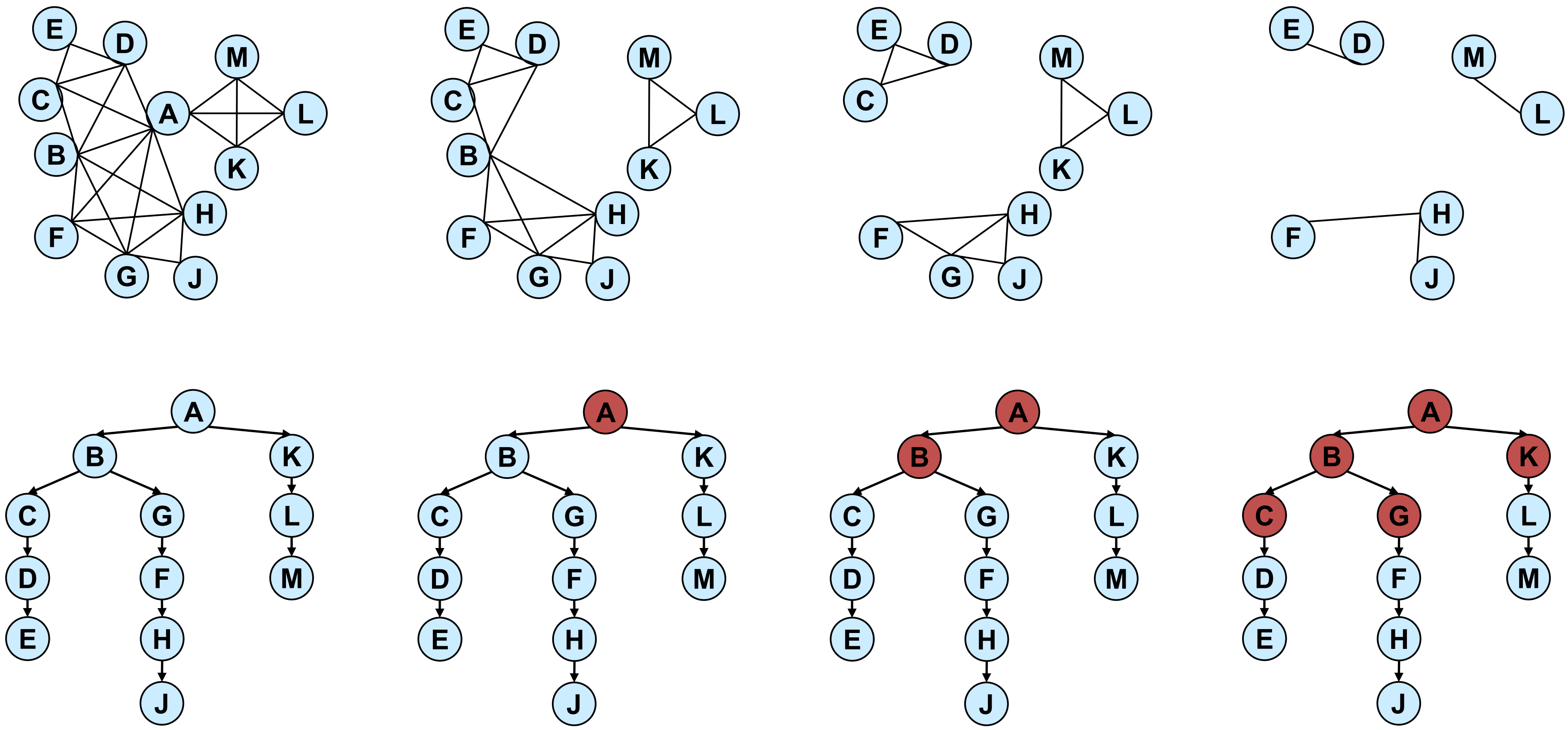
Graph Coloring problem



- Inference may require too much memory
- **Condition** on some of the variables



AND/OR w-cutset



3-cutset 2-cutset 1-cutset

Summary: Search methods

- **AND/OR search spaces** exploit the structure of the graphical model and create a far more compact search space.
  - **AND/OR Trees** are exp(height) of the pseudo-tree and can be traversed in linear memory
  - **AND/OR Graphs** are exp(induced-width) of the pseudo-tree and require exp(w) memory when traversed.
  - **The pseudo-tree structure** is instrumental in facilitating effective search

- **The MBE heuristic** can guide heuristic search (depth-first, Best-first or hybrid) pruning search further.
- **Tasks:** this schemes are applicable to a large class of tasks:
  - Belief updating, marginal map and Influence diagram search
- **Mixed schemes of Inference and Search** like Cutset schemes facilitate tradeoff between memory and time.