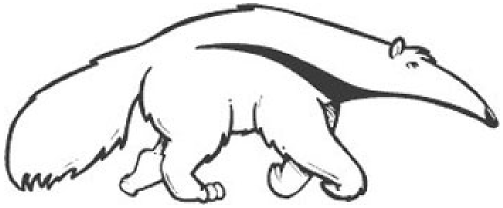
Algorithms for Causal Probabilistic Graphical Models

Class 5:

Causal Queries & Observational Data

Athens Summer School on AI
July 2024



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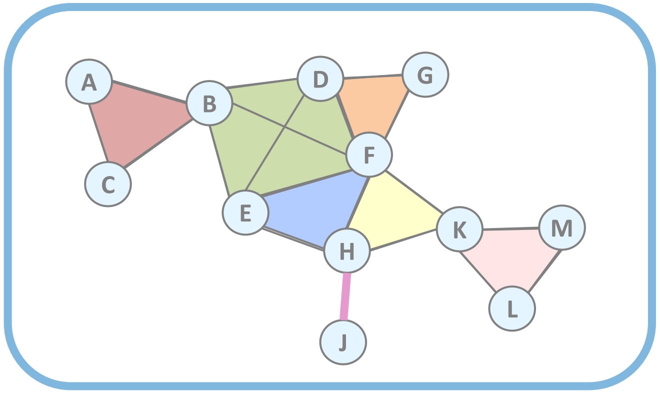
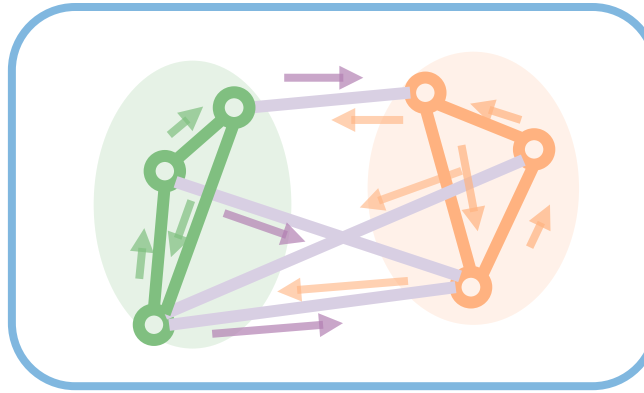
Outline of Lectures

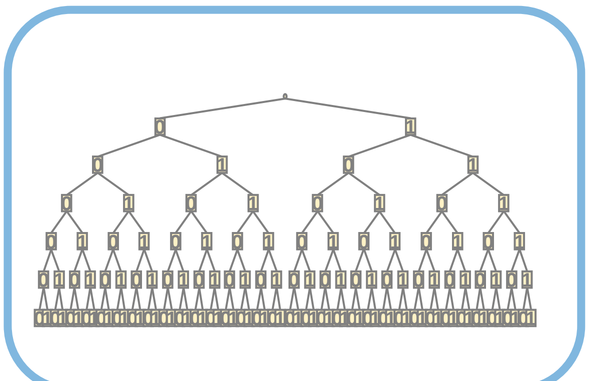
Class 1: Introduction & Inference



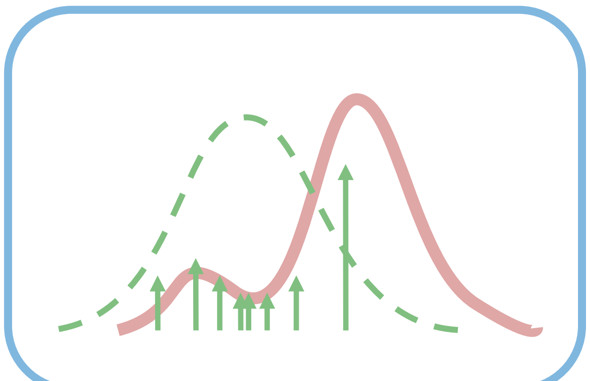
Class 2: Bounds & Variational Methods



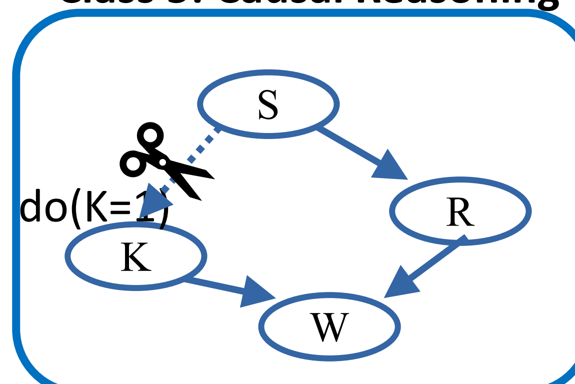
Class 3: Search Methods



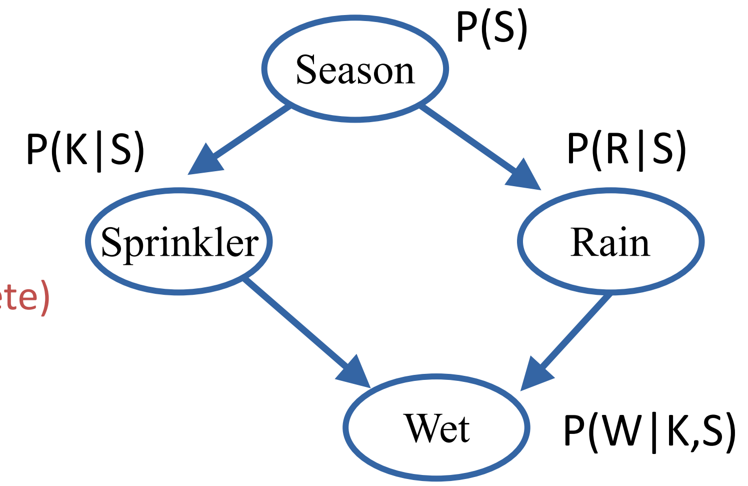
Class 4: Monte Carlo Methods



Class 5: Causal Reasoning



Graphical models



A *graphical model* consists of:

$X = \{X_1, \ldots, X_n\}$ -- variables

$D = \{D_1, \ldots, D_n\}$ -- domains (we'll assume discrete)

$F = \{f_{\alpha_1}, \ldots, f_{\alpha_m}\}$ -- functions or CPTs

and a *combination operator*

The *combination operator* defines an overall function from the individual factors,

e.g., "+" : $P(S,K,R,W) = P(S) \cdot P(K|S) \cdot P(R|S) \cdot P(W|K,S)$

Notation:

Discrete X_i values called "states"

"Tuple" or "configuration": states taken by a set of variables

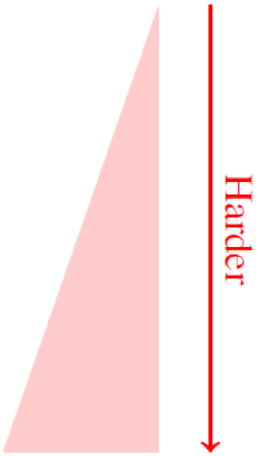
"Scope" of f: set of variables that are arguments to a factor f

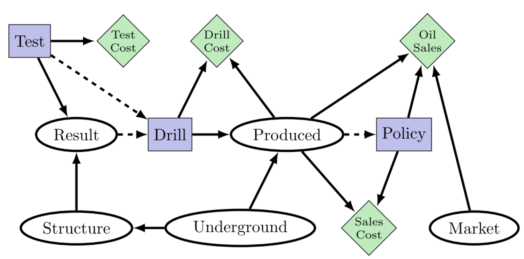
often index factors by their scope, e.g., $f_\alpha(X_\alpha), \quad X_\alpha \subseteq X$

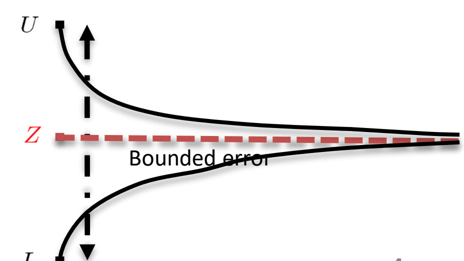
Probabilistic Reasoning Problems

- Exact inference time, space exponential in induced width
- Casual reasoning is a sum-inference task.

Max-Inference:	$f(x^*) = \max_x \prod_\alpha f_\alpha(x_\alpha)$
Sum-Inference: (e.g., causal effects)	$Z = \sum_x \prod_\alpha f_\alpha(x_\alpha)$
Mixed-Inference (MMAP):	$f_M(x_M^*) = \max_{x_M} \sum_{x_S} \prod_\alpha f_\alpha(x_\alpha)$
Mixed-Inference (MEU): (e.g., decisions, planning)	$\text{MEU} = \max_{D_1,\ldots,D_m} \sum_{X_1,\ldots X_n} (\prod_{P_i \in P} P_i) \times (\sum_{r_i \in R} r_i)$







Outline: Causal Inference

- Causal Models: Semantics
- **Causal Models: Queries**
- Identifiability
- Estimand Methods
- Learning Methods

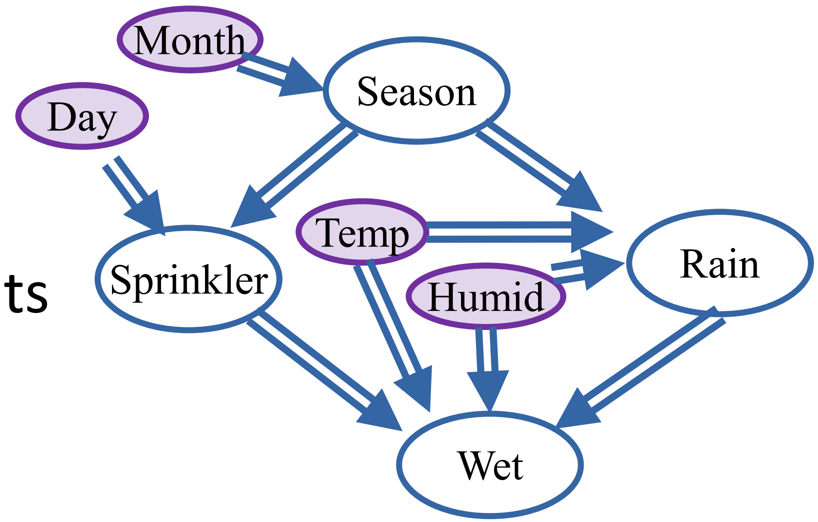
Pearl's Causal Hierarchy (PCH)

Level (Symbol)	Typical Activity	Typical Question	Examples
1 👀 Association $P(y \mid x)$	Seeing	What is? How would seeing X change my belief in Y?	What does a symptom tell us about the disease?
2 💪 Intervention $P(y \mid do(x), c)$	Doing	What if? What if I do X?	What if I take aspirin, will my headache be cured?
3 🧠 Counterfactual $P(y_x \mid x', y')$	Imagining, Retrospection	Why? What if I had acted differently?	Was it the aspirin that stopped my headache?

Structural Causal Models

- Endogenous (visible) variables V
  - Season, Sprinkler, Rain, Wet…
- Exogenous (latent) variables U
  - Temp, Humidity, Day, Month
- V are deterministic ($\Rightarrow$) given parents
  - $v_i = f_i(pa_i, u_i)$
- Randomness arises from U
  - $(u_1, \ldots, u_m) \sim p(U)$

Ex: Sprinkler



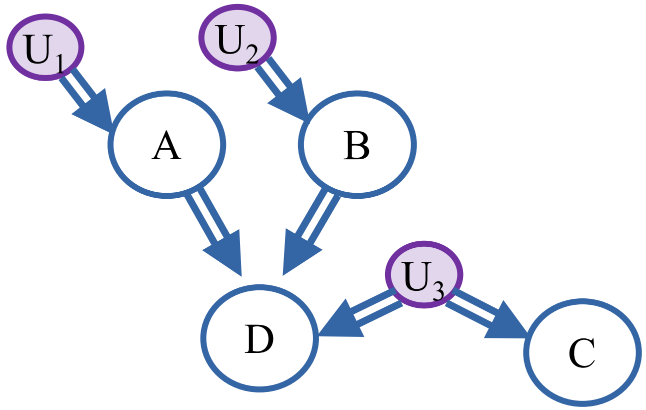
- We can only observe the variables V
  - SCM defines a **causal diagram**
    and the **observational distribution** p(V)

$$p(V) = \sum_{\mathbf{u}} p(\mathbf{u}) \prod_i p(V_i \,|\, pa_i, u_i)$$

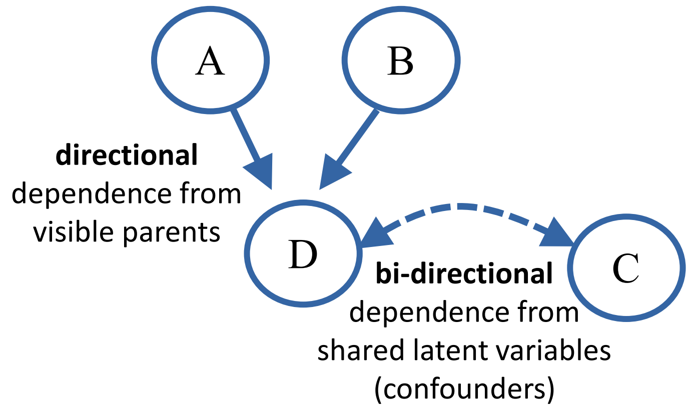
Causal Diagram

A graph over the visible variables V that describes their causal structure

Structural Causal Model



Causal Diagram



Special Cases

Markovian

- Each U_i has no parents, one child
  (equivalent to a Bayesian network)

Semi-Markovian

- Each U_i has no parents, ≤ 2 children

Observational Distribution

$$p(V) = \sum_{\mathbf{u}} p(\mathbf{u}) \prod_i p(V_i \,|\, pa_i, u_i)$$

visible and latent parents of V_i

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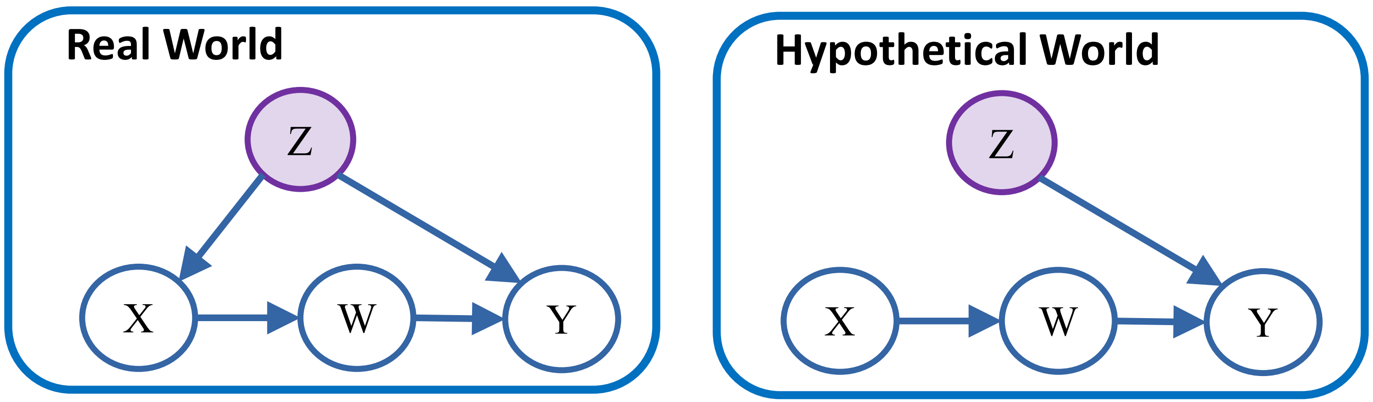
The Challenge of Causal Inference

"Causal Effect"

- How much does outcome Y change with X, if we vary X between two constants free of the influence of other (possibly unobserved) causes Z?



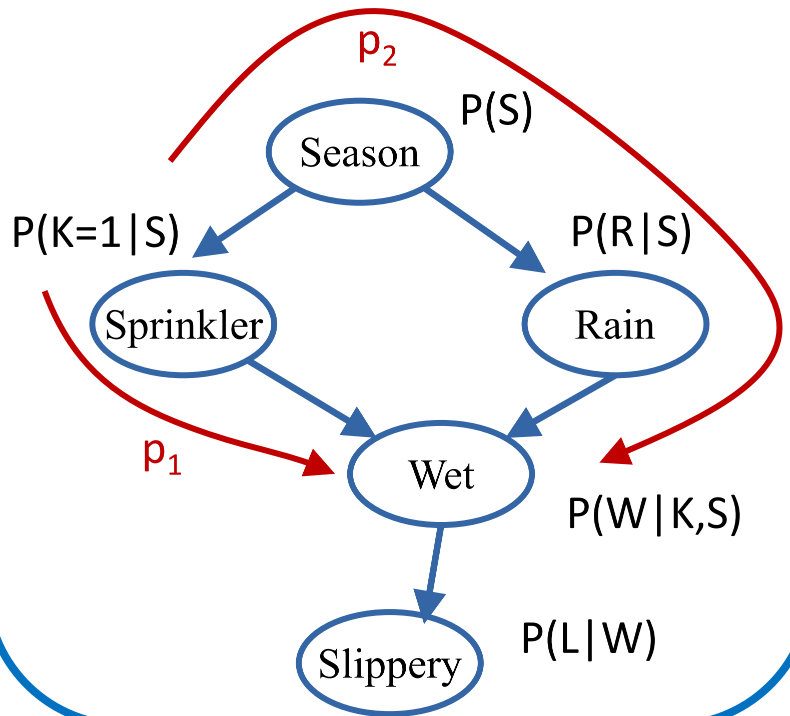
Z : age, sex
X : action
W : mediator
Y : outcome

- Randomized control experiments
  - Sample from hypothetical world directly
  - What if we cannot do this? (e.g., can't control X directly, or too much delay)
- Can we estimate using data only from the left model?

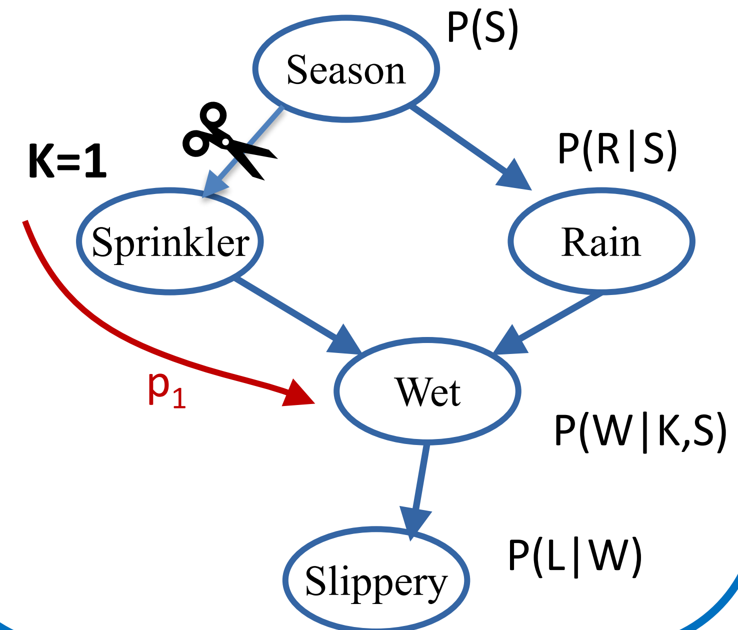
Computing Causal Effects

$$p(W|K=1) = ?$$

$$= \frac{\sum_{S,R} p(W|K=1,R)\, p(K=1|S)\, p(R|S)\, P(S)}{\sum_{S} p(K=1|S)\, p(S)}$$



$$p(W|\mathrm{do}(K=1)) = ?$$



Computing Causal Effects

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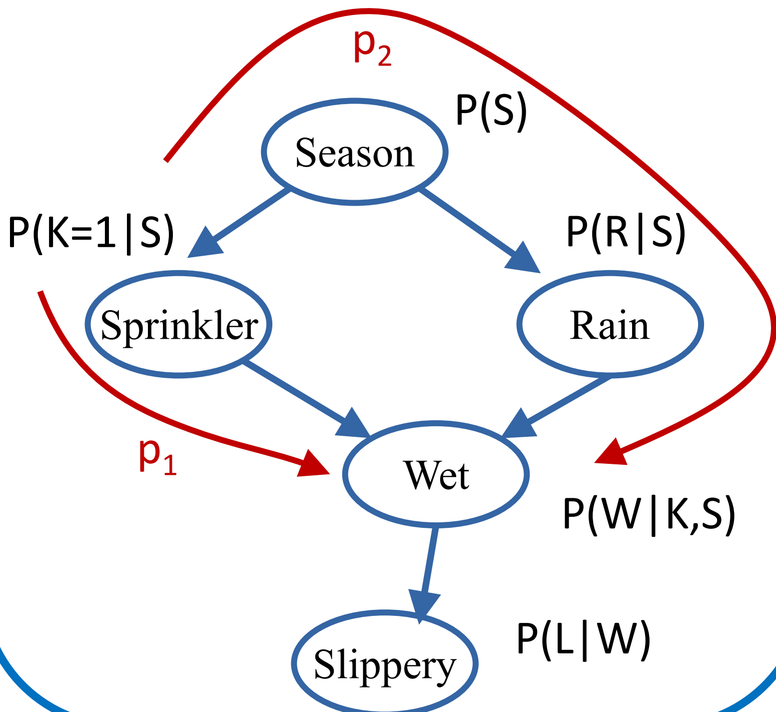
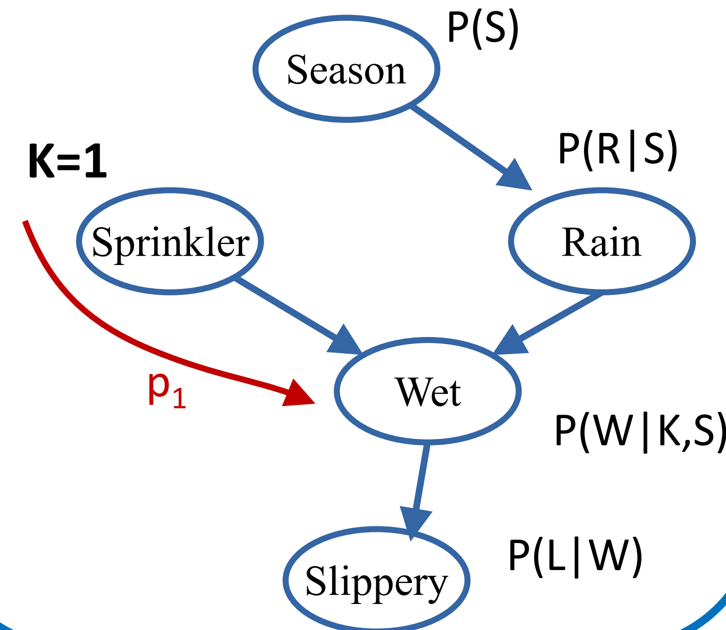


$$p(W|\mathrm{do}(K=1)) = ?$$

$$= \frac{\sum_{S,R} p(W|K=1,R)\, \mathbb{1}(K=1)\, p(R|S)\, P(S)}{\mathbb{1}(K=1)}$$



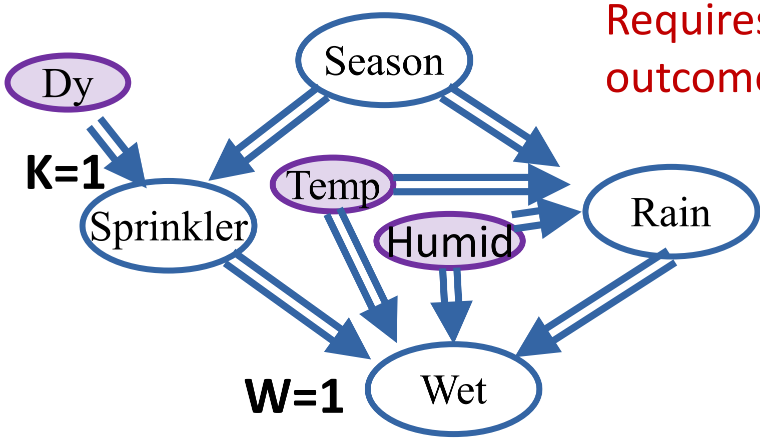
Counterfactual Queries

Counterfactual Query:

Probability of an event in contradiction with the observations

*What **would have happened** if the sprinkler had been turned off?*

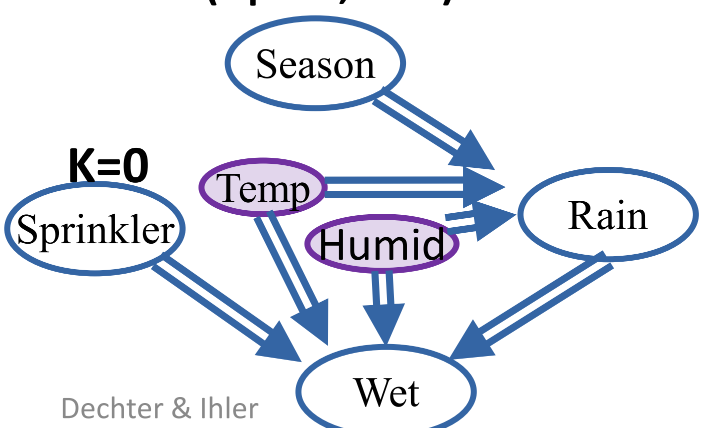
Requires that we transfer information about random outcomes that happened, to a different setting



Observe the sprinkler is on & grass is wet: (K=1,W=1)

What is the probability it would still be wet if we had turned the sprinkler off?

P(S|K=1,W=1)



Abduction: Observing K=1 tells us it is more likely to be summer;
Observing K=1,W=1 tells us it is not too hot & dry.

Action and Prediction: Then, apply this knowledge to compute the counterfactual:

Computing Counterfactuals

Given a model $M=\langle \mathbf{V},\mathbf{U},F,P(\mathbf{u})\rangle$, the conditional probability $P(\mathbf{Y}_x \mid \mathbf{z})$ can be evaluated using the following 3-step procedure:

1. (Abduction) Update $P(\mathbf{u})$ by the evidence $\mathbf{Z}=\mathbf{z}$ to obtain $P(\mathbf{u} \mid \mathbf{z})$.
2. (Action) Modify M with $do(\mathbf{X}=\mathbf{x})$ to obtain $F_{\mathbf{x}}$.
3. (Prediction) Use the model $\langle \mathbf{V},\mathbf{U},F_{\mathbf{x}},P(\mathbf{u} \mid \mathbf{z})\rangle$ to compute the probability of $\mathbf{Y}$.

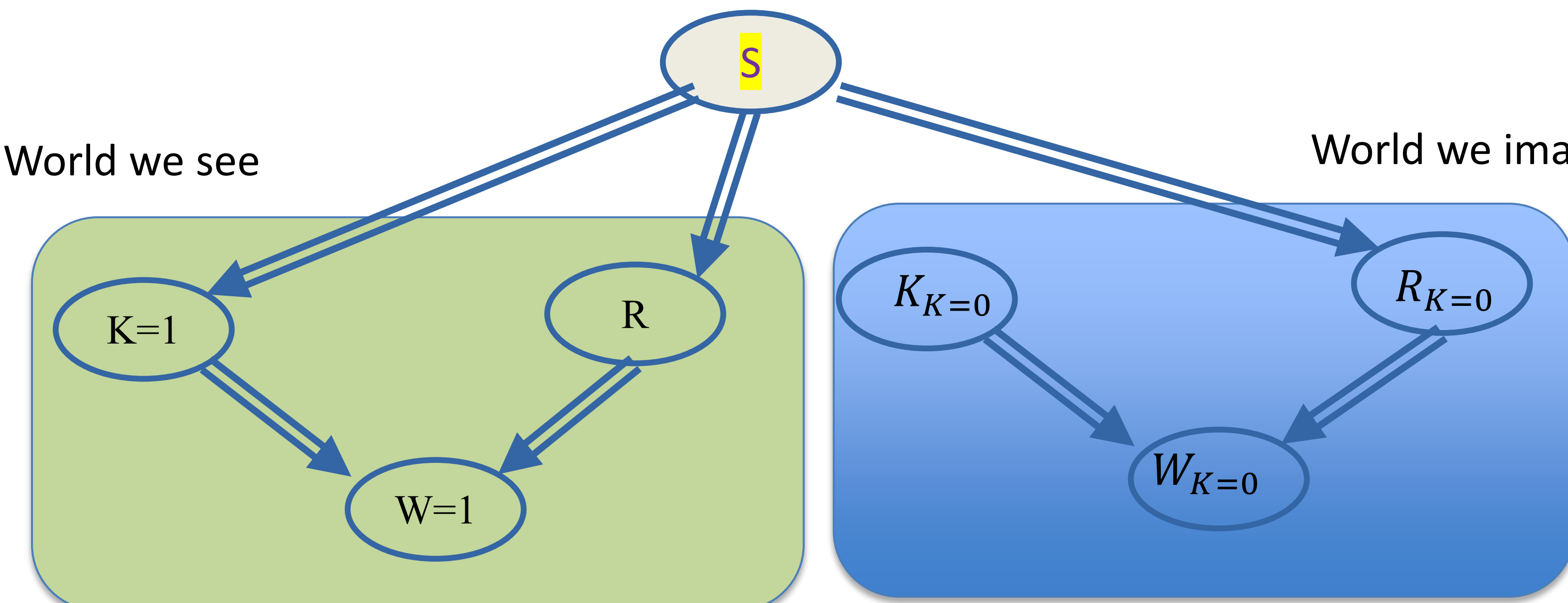
Counterfactual Queries

Ex: Observe the sprinkler is on & grass is wet: (K=1,W=1). What is the probability it would still be wet if we had turned the sprinkler off? Observing K=1 tells us it is

If we have the full model, Counterfactual queries can
Be answered by PGM methods over the twin network model (Classes 1-4)

ph G*

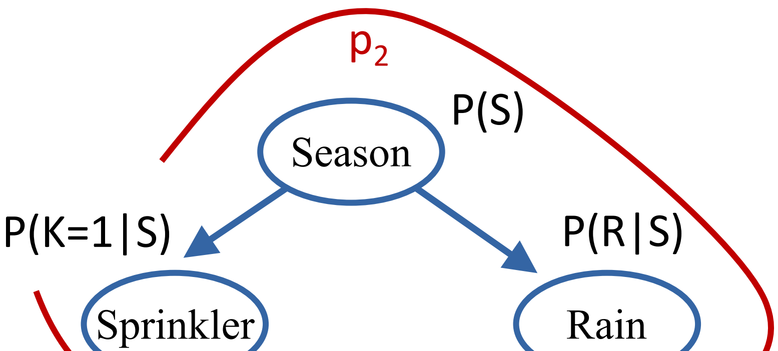
Compute $p(W_{K=0} \mid K=1, W=1)$ in M*



Computing Causal Effects

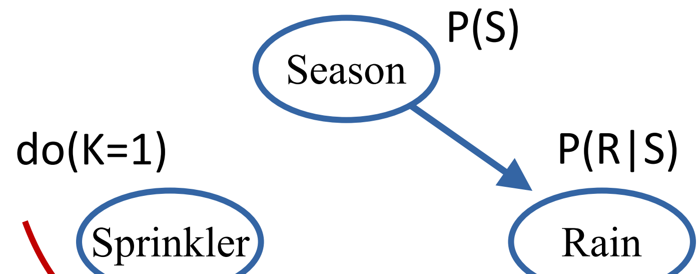
$$p(W|K=1) = ?$$

$$= \frac{\sum_{S,R} p(W|K=1,R)\, p(K=1|S)\, p(R|S)\, P(S)}{\sum_S p(K=1|S)\, p(S)}$$



$$p(W|\mathrm{do}(K=1)) = ?$$

$$= \frac{\sum_{S,R} p(W|K=1,R)\, \mathbb{1}(K=1)\, p(R|S)\, P(S)}{\mathbb{1}(K=1)}$$



If the model is **known**:

- Causal effects and counterfactual queries can be computed using inference
- (classes 1-4)

What if model is **unknown**?

- When is it possible to estimate the causal effect from observed data?
- When possible, how can we do it?

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- **Identifiability**
- Estimand Methods
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Identifiability

- When can we answer p(Y|do(X)) from observations?

Definition

We say a query p(Y|do(X)) is **identifiable** on graph G if, for any two distributions $p_1(V,U)$, $p_2(V,U)$ on G,

$$p_1(V) = p_2(V) \quad \Rightarrow \quad p_1(Y|do(X)) = p_2(Y|do(X))$$

- Intuition
  - If a query is not identifiable, it cannot be answered uniquely for **any** amount of data – no consistent estimator exists!
  - Conversely, if we can express p(Y|do(X)) in terms of p(V), the query must be identifiable.

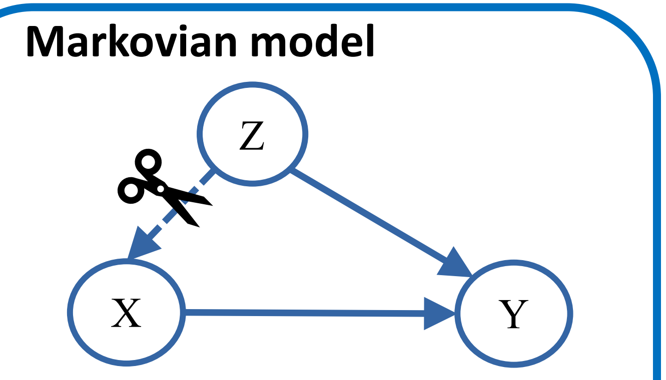
Let's look at a few useful special cases, before the general setting…

Identifiability: Markovian models

- For a Markovian graph G:
  - Causal effect p(Y|do(X)) is identifiable whenever X and all its parents are observed
  - In general,

$$p(Y|\mathrm{do}(X)) = \sum_Z p(Y|X, pa_X)\, p(pa_X)$$

We “adjust” for the values of pa_X!



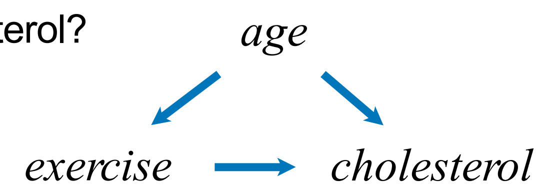
$$p(Y|\mathrm{do}(X)) = \sum_Z p(Y|X, Z)\, p(Z)$$

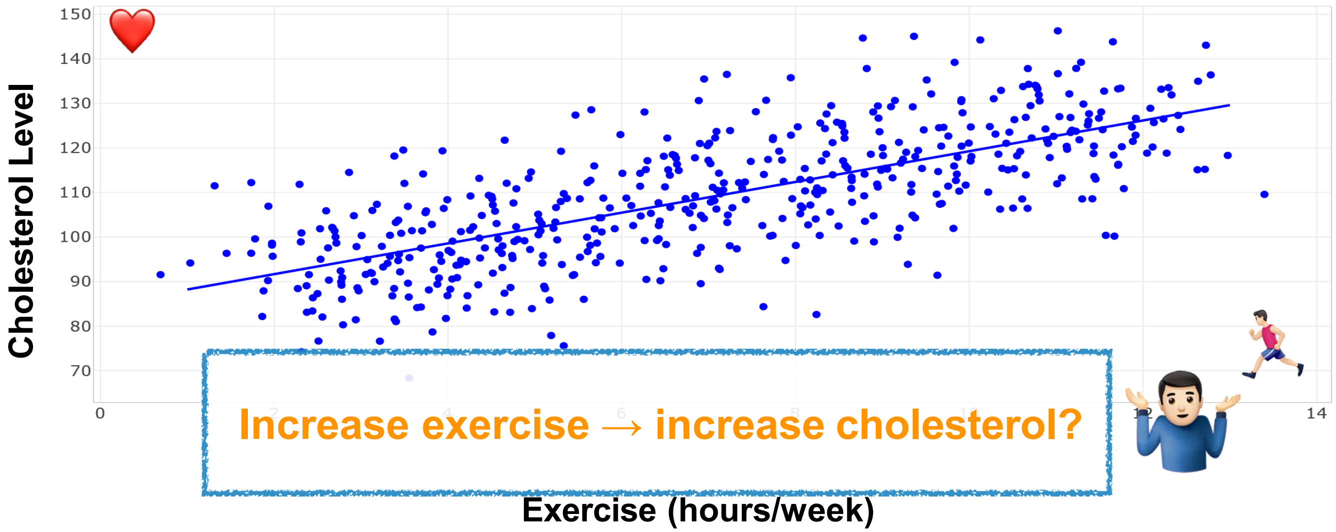
- Why is this necessary?
  - The problem of confounding

Ex: Confounding Bias

What's the causal effect of Exercise on Cholesterol?

What about $P(cholesterol \mid exercise)$?

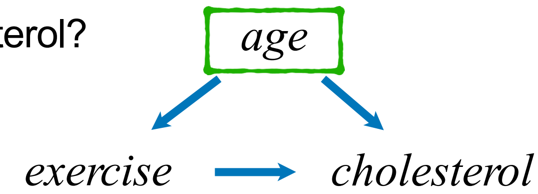


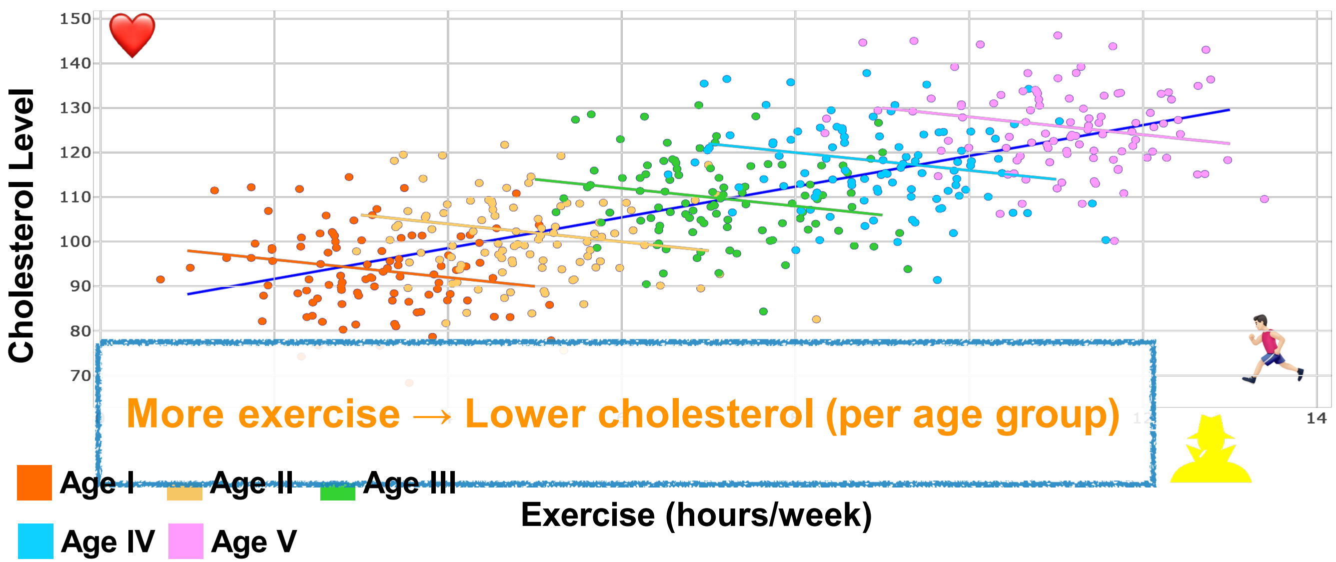


Ex: Confounding Bias

What's the causal effect of Exercise on Cholesterol?

What about *P(cholesterol | exercise)* ?





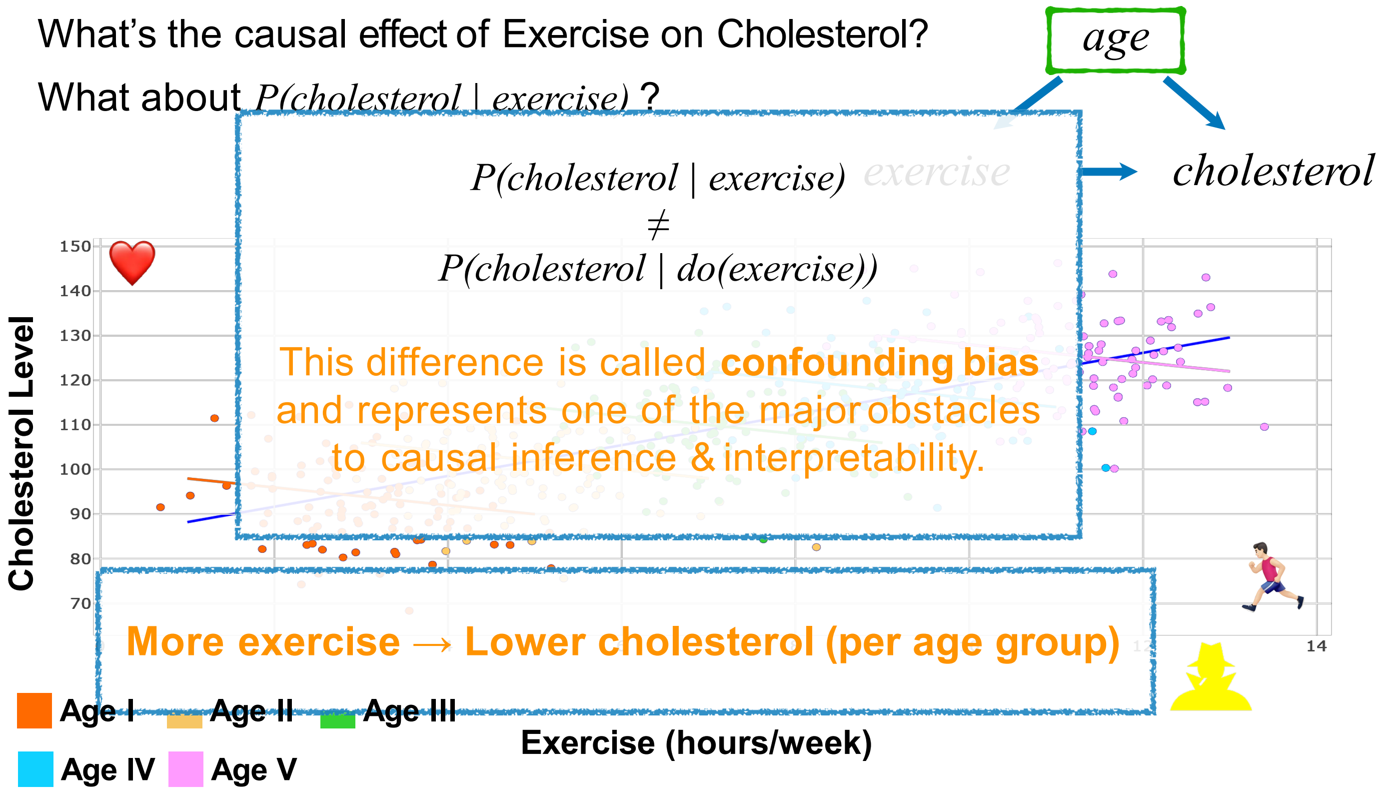
Ex: Confounding Bias

What's the causal effect of Exercise on Cholesterol?

What about *P(cholesterol | exercise)* ?

$$P(cholesterol \mid exercise) \neq P(cholesterol \mid do(exercise))$$

This difference is called **confounding bias** and represents one of the major obstacles to causal inference & interpretability.



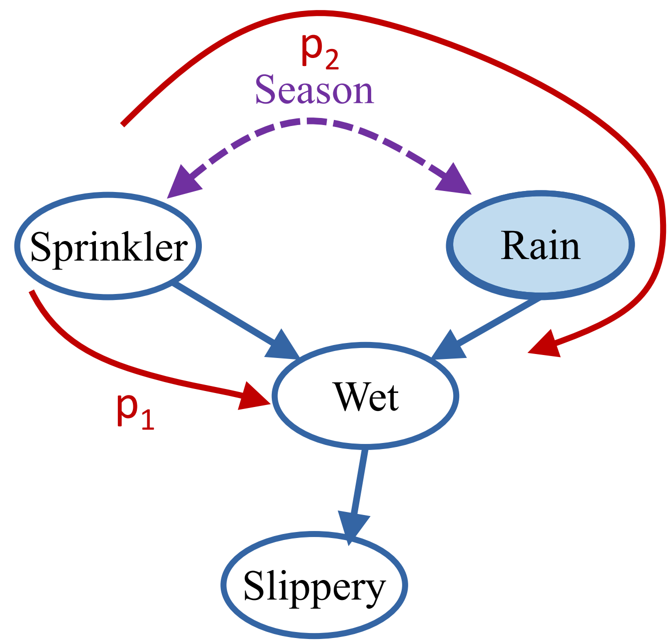
More exercise → Lower cholesterol (per age group)

Identifiability: Backdoor Criterion

- A set Z satisifies the **backdoor criterion** if
  - No Z_i in Z is a descendant of X
  - Z blocks every path between X,Y that has an arrow into X
- Then, $p(Y|\text{do}(X)) = \sum_{Z} p(Y|X,Z)\, p(Z)$

Ex: What if Season is latent?

- Z={Rain} for X=Sprinkler, Y=Wet
  - Conditioning on Rain blocks the non-causal path p_2
  - Leaves the causal path p_1 unaffected!

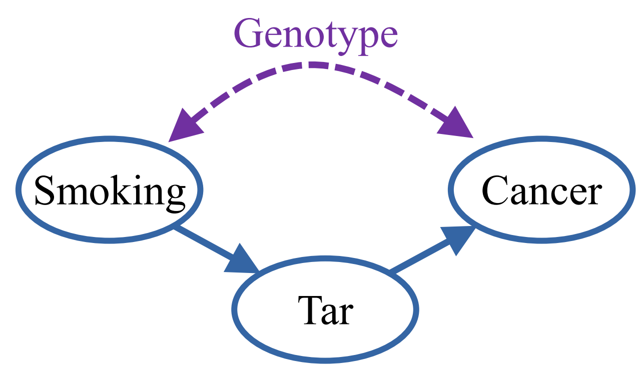


Identifiability: Frontdoor Criterion

- A set Z satisifies the **frontdoor criterion** if
  - Z intercepts all directed paths from X to Y
  - There is no unblocked backdoor path from X to Z
  - All backdoor paths from Z to Y are blocked by X
- Then, $p(Y|\text{do}(X)) = \sum_{Z} p(Z|X) \sum_{X'} p(Y|X', Z)\, p(X')$

Ex: Smoking & Cancer

- Z={Tar} for X=Smoking, Y=Cancer



$$p(Y|\text{do}(X)) = \sum_{Z} p(Z|X) \sum_{X'} p(Y|X', Z)\, p(X')$$

p(Z|do(X)) p(Y|do(Z))

”mediating variable”
in causation process

The Do-Calculus

- Semantics for rewriting expressions with do-operators

Theorem

The following transformations are valid for any do-distribution induced by a causal model M:

Rule 1: Adding/Removing Observations

$$p(y|\text{do}(x), \text{do}(z), w) = p(y|\text{do}(x), z, w) \qquad \text{if} \quad (Z \perp\!\!\!\perp Y \mid X, W)_{G_{\overline{X}\underline{Z}}}$$

Rule 2: Action/Observation Exchange

$$p(y|\text{do}(x), z, w) = p(y|\text{do}(x), w) \qquad \text{if} \quad (Z \perp\!\!\!\perp Y \mid W)_{G_{\overline{X}}}$$

Rule 3: Adding/Removing Actions

$$p(y|\text{do}(x), \text{do}(z), w) = p(y|\text{do}(x), w) \qquad \text{if} \quad (Z \perp\!\!\!\perp Y \mid X, W)_{G_{\overline{X}\,\overline{Z(W)}}}$$

where Z(W) is the set of Z-nodes that are not ancestors of any W-node in $G_{\overline{X}}$

If we can rewrite p(Y|do(X)) in terms of p(V), the query is identifiable!

Algorithmic approach for identification

Truncated Product in Semi-Markovian Models

The distribution generated by an intervention $do(\boldsymbol{X}=\boldsymbol{x})$ in a Semi-Markovian model M is given by the (generalized) truncated factorization product, namely,

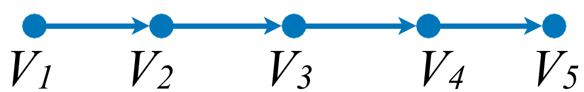
$$P(\mathbf{v} \mid do(\mathbf{x})) = \sum_{\mathbf{u}} \prod_{\{V_i \in \mathbf{V} \setminus \mathbf{X}\}} P\left(v_i \mid pa_i, u_i\right) P(\mathbf{u})$$

And the effect of such intervention on a set $\boldsymbol{Y}$ is

$$P(\mathbf{y} \mid do(\mathbf{x})) = \sum_{\mathbf{v} \setminus (\mathbf{y} \cup \mathbf{x})} \sum_{\mathbf{u}} \prod_{\{V_i \in \mathbf{V} \setminus \mathbf{X}\}} P\left(v_i \mid pa_i, u_i\right) P(\mathbf{u})$$

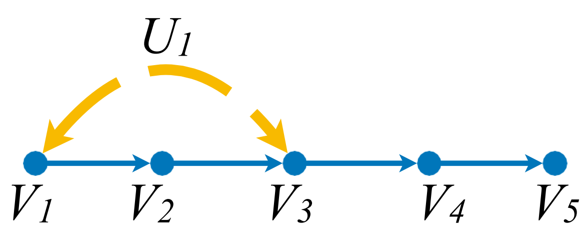
Factorizing the observed distribution

- Start from a simple Markovian model:



$$P(\mathbf{v}) = P(v_1)P(v_2 \mid v_1)P(v_3 \mid v_2)P(v_4 \mid v_3)P(v_5 \mid v_4)$$

- Let's add an unobservable U_1, that affects two observables, and breaking Markovianity:

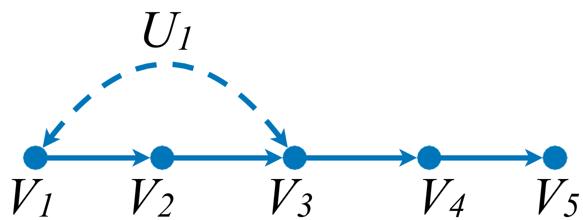


$$P(\mathbf{v}) = \sum_{u_1} P(u_1)\underline{P(v_1 \mid u_1)}P(v_2 \mid v_1)\underline{P(v_3 \mid v_2, u_1)}P(v_4 \mid v_3)P(v_5 \mid v_4)$$

$$= P(v_2 \mid v_1)P(v_4 \mid v_3)P(v_5 \mid v_4)\left(\underline{\sum_{u_1} P(u_1)P(v_1 \mid u_1)P(v_3 \mid v_2, u_1)} \right)$$

Factorizing the observed distribution

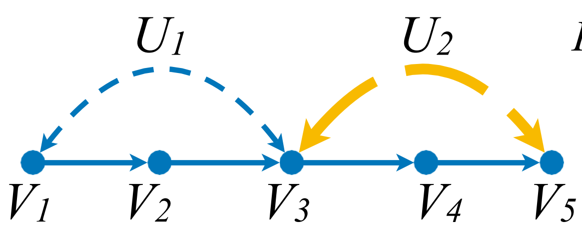
- From the previous model …



$$P(\mathbf{v}) = \sum_{u_1} P(u_1)P(v_1 \mid u_1)P(v_2 \mid v_1)P(v_3 \mid v_2, u_1)P(v_4 \mid v_3)P(v_5 \mid v_4)$$

$$= P(v_2 \mid v_1)P(v_4 \mid v_3)P(v_5 \mid v_4)\left(\sum_{u_1} P(u_1)P(v_1 \mid u_1)P(v_3 \mid v_2, u_1) \right)$$

- Add another unobservable U_2,

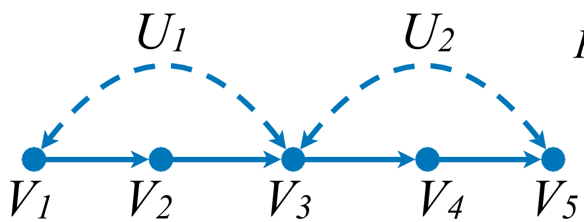


$$P(\mathbf{v}) = \sum_{u_1, u_2} P(u_1, u_2)P(v_1 \mid u_1)P(v_2 \mid v_1)P(v_3 \mid v_2, u_1, u_2)P(v_4 \mid v_3)P(v_5 \mid v_4, u_2)$$

$$= P(v_2 \mid v_1)P(v_4 \mid v_3)\left(\sum_{u_1, u_2} P(u_1, u_2)P(v_1 \mid u_1)P(v_3 \mid v_2, u_1, u_2)P(v_5 \mid v_4, u_2) \right)$$

Factorizing the observed distribution

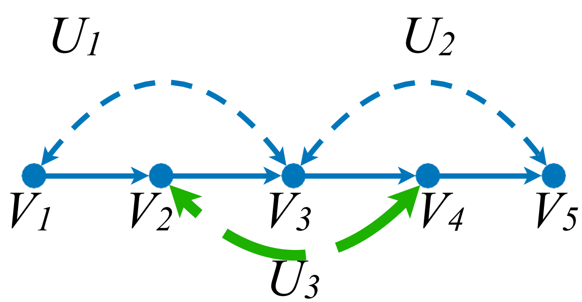
- From the previous model…



$$P(\mathbf{v}) = \sum_{u_1,u_2} P(u_1,u_2)P(v_1 \mid u_1)P(v_2 \mid v_1)P(v_3 \mid v_2,u_1,u_2)P(v_4 \mid v_3)P(v_5 \mid v_4,u_2)$$

$$= P(v_2 \mid v_1)P(v_4 \mid v_3)\left(\sum_{u_1,u_2} P(u_1,u_2)P(v_1 \mid u_1)P(v_3 \mid v_2,u_1,u_2)P(v_5 \mid v_4,u_2)\right)$$

- Let's add one more, U_3,

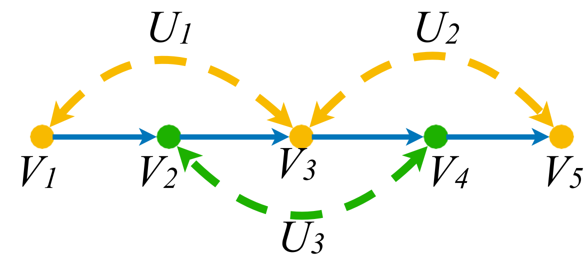


$$P(\mathbf{v}) = \sum_{u_1,u_2,u_3} P(u_1,u_2,u_3)P(v_1 \mid u_1)P(v_2 \mid v_1,u_3)P(v_3 \mid v_2,u_1,u_2)$$
$$P(v_4 \mid v_3,u_3)P(v_5 \mid v_4,u_2)$$

$$= \left(\sum_{u_3} P(u_3)P(v_2 \mid v_1,u_3)P(v_4 \mid v_3,u_3)\right)\left(\sum_{u_1,u_2} P(u_1,u_2)P(v_1 \mid u_1)P(v_3 \mid v_2,u_1,u_2)P(v_5 \mid v_4,u_2)\right)$$

C-Factors

- Recall our example



$$P(\mathbf{v}) = \left(\underbrace{\sum_{u_3} P(u_3)P(v_2 \mid v_1, u_3)P(v_4 \mid v_3, u_3)} \right) \left(\underbrace{\sum_{u_1,u_2} P(u_1, u_2)P(v_1 \mid u_1)P(v_3 \mid v_2, u_1, u_2)P(v_5 \mid v_4, u_2)} \right)$$

- These factors made of sums may be long to write in terms of $P(\mathbf{v},\mathbf{u})$. However, their structure follows from the topology of the diagram, then we can abstract this concept out by defining a new function Q:

$$Q[\mathbf{C}](\mathbf{c}, pa_{\mathbf{c}}) = \sum_{u(\mathbf{C})} P(u(\mathbf{C})) \prod_{V_i \in \mathbf{C}} P(v_i \mid pa_i, u_i) \qquad \text{where} \qquad U(\mathbf{C}) = \bigcup_{V_i \in \mathbf{C}} U_i$$

Then $P(\mathbf{v})$ can be re-written as

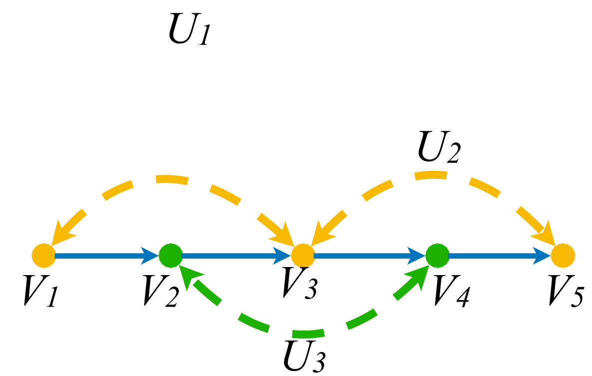
$$P(\mathbf{v}) = \underbrace{Q[V_2, V_4](v_2, v_4, v_1, v_3)}\underbrace{Q[V_1, V_3, V_5](v_1, v_3, v_5, v_2, v_4)}$$

C-Factors

- For convenience $Q[\mathbf{C}](\mathbf{c},pa_c)$ can be written just as $Q[\mathbf{C}]$

- Then, for our example, we can just write

$$P(\mathbf{v}) = Q[V_2, V_4]Q[V_1, V_3, V_5]$$



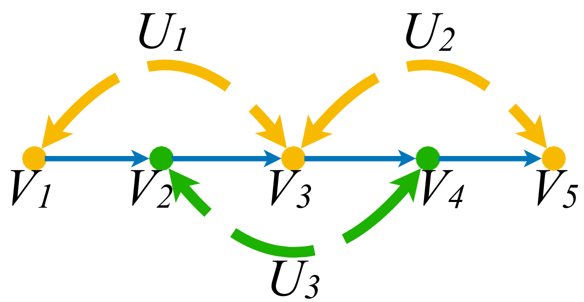
- Note that for the whole set of variables V

$$Q[\mathbf{V}] = \sum_{\mathbf{u}} P(\mathbf{u}) \prod_{V_i \in \mathbf{V}} P(v_i \mid pa_i, u_i) = P(\mathbf{v})$$

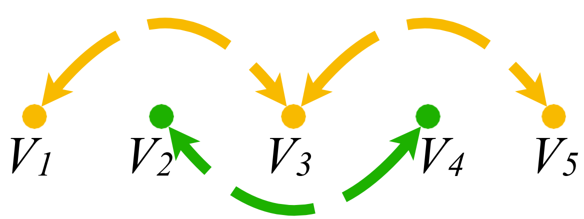
- For consistency define $Q[\varnothing]=1$

Confounding Components

C-components: A partition of the observed variables where any 2 variables connected by a path of bi-directed edges is in the same component.



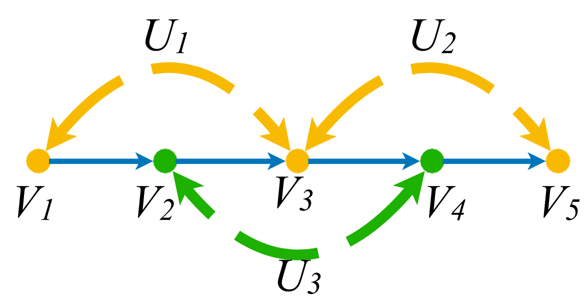
- V_1 is in the same c-component as V_3,
- V_3 is in the same c-component as V_5,
- By extension, V_1 is in the same c-component as V_5 too.
- V_2 is in the same c-component as V_4.



- To see it easily, consider the graph induced over the bidirected edges!
- Obs. The C-Component relation defines a partition over the observable variables, hence it is *Reflexive*, *Symmetric* and *Transitive*.

C-Component Factorization

- The distribution $P(\mathbf{v})$ factorizes into c-factors associated with the c-components of the graph.



$$Q_1 = \{V_2, V_4\} \quad Q_2 = \{V_1, V_3, V_5\}$$

$$P(\mathbf{v}) = \left(\sum_{u_3} P(u_3) P(v_2 \mid v_1, u_3) P(v_4 \mid v_3, u_3) \right) \left(\sum_{u_1, u_2} P(u_1, u_2) P(v_1 \mid u_1) P(v_3 \mid v_2, u_1, u_2) P(v_5 \mid v_4, u_2) \right)$$

$$P(\mathbf{v}) = Q[V_2, V_4]\, Q[V_1, V_3, V_5]$$

C-Component Factorization

- For any $\boldsymbol{H} \subseteq \boldsymbol{V}$, consider a graph $G_{\boldsymbol{H}}$.
- Let $H_1, H_2, \ldots, H_k$ be the c-components of $G_{\boldsymbol{H}}$.
- Then

$$Q[\mathbf{H}] = \prod_j Q[H_j]$$

And, the Q factor of any c component can be computed from Q(H)

C-factor Algebra - Summary

We have two basic operations over c-factors

1. Reduce to an ancestral set

$$Q[\mathbf{W}] = \sum_{\mathbf{c}\setminus\mathbf{w}} Q[\mathbf{C}]$$

If $\boldsymbol{W}$ is ancestral in $G_{\boldsymbol{C}}$

2. Factorize into c-components

$$Q[\mathbf{H}] = \prod_{j} Q[H_j]$$

Where $H_1, \ldots H_k$, are the c-components in $G_{\boldsymbol{H}}$

The Identification Algorithm

- Given G and the query variables $\boldsymbol{X}, \boldsymbol{Y}$

$$P(\mathbf{y} \mid do(\mathbf{x})) = \sum_{\mathbf{v} \setminus (\mathbf{x} \cup \mathbf{y})} Q[\mathbf{V} \setminus \mathbf{X}]$$

$$= \sum_{\boldsymbol{d} \setminus \mathrm{y}} Q[\boldsymbol{D}]$$

where $\boldsymbol{D} = An(\boldsymbol{Y})$ in $G_{\overline{\mathbf{X}}}$

- Suppose the graph $G_{\boldsymbol{D}}$ has C-components $\mathbf{D}_1, \mathbf{D}_2, \ldots, \mathbf{D}_k$, then

$$P(\mathbf{y} \mid do(\mathbf{x})) = \sum_{\mathbf{d} \setminus \mathbf{y}} \prod_{i} Q[\mathbf{D}_i]$$

The Identification Algorithm

- If we can identify each $Q[D1], Q[D2], \ldots, Q[Dk]$, from $P(\mathbf{v}) = Q[V] = Q[C_1] \ldots Q[C_m]$, we obtain an expression equal to $P(\mathbf{y}|do(\mathbf{x}))$. An algorithm for computing Q[C] from Q[T] for C, T being c-components:

```
Identify(C, T, Q, G)
1. Let A=An(C) in G_T.
2. If A=C return Q[C]=Σ_{t\c}Q.
3. If A=T return Fail.
4. Let A_i be the c-comp of G_A that contains C.
   Get Q[A_i] from Q.
5. Return Identify(C, A_i, Q[A_i], G).
```

Completeness

Theorem [Huang and Valtorta, 2008]

The causal effect $P(\mathbf{y}|do(\mathbf{x}))$ is identifiable from causal diagram G and $P(\mathbf{v})$ *if and only if* each of the C-factors $Q[\mathbf{D_i}$ is identifiable by `Identify(Di,Ci,Q[Ci],G)`.

Where $\mathbf{C}_i$ is the C-component of G containing $\mathbf{D}_i$.

Examples of Estimand Expressions

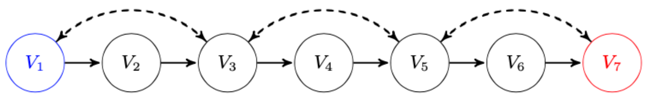
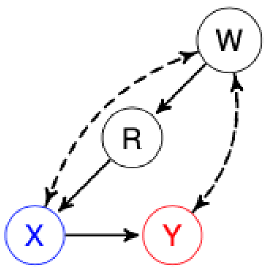


Figure 1: Chain Model with 7 observable variables and 3 latent variables

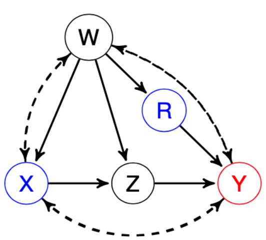
$$P(V_7 \mid do(V_1)) = \sum_{V_2,V_3,V_4,V_5,V_6} P(V_6 \mid V_1,V_2,V_3,V_4,V_5)P(V_4 \mid V_1,V_2,V_3)P(V_2 \mid V_1)$$
$$\times \sum_{V_1'} P(V_7 \mid V_1',V_2,V_3,V_4,V_5,V_6)P(V_5 \mid V_1',V_2,V_3,V_4)P(V_3 \mid V_1',V_2)P(V_1')$$



(a) Model 1

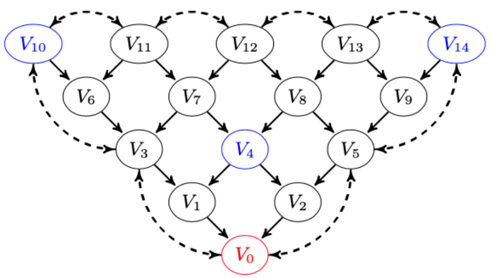
Estimand Expressions for Models 1 & 8 .

Model	Estimate of $P(Y \mid do(X))$
1	$\frac{\sum_W P(X,Y \mid R,W)P(W)}{\sum_W P(X \mid R,W)P(W)}$
8	$\sum_{R,W,Z} P(Z \mid R,W,X)P(R \mid W) \sum_x P(Y \mid R,W,x,Z)P(x \mid R,W)P(W)$



(c) Model 8

Examples of Estimand Expressions



(b) Cone Cloud, $n = 15$ (15-CC)

$$
\begin{aligned}
P(V_0|V_{14}, V_{10}, V_4) = & \\
\sum_{V_1,V_2,V_3,V_5,V_6,V_7,V_8,V_9,V_{11},V_{12},V_{13},v'_{10},V'_{11},V'_{12},V'_{13},V'_{14}} & P(V_2|V_4, V_5, V_7, V_8, V_9, V_{11}, V_{12}, V_{13}, V_{14})\times \\
& P(V_9|V_{13}, V_{14})P(V_8|V_{12}, V_{13})P(V_1|V_3, V_4, V_6, V_7, V_8, V_{10}, V_{11}, V_{12}, V_{13})\times \\
& P(V_7|V_{11}, V_{12})P(V_6|V_{10}, V_{11})P(V_{11}, V_{12}, V_{13})\times \\
& P(V_0|V_1, V_2, V_3, V_4, V_5, V_6, V_7, V_8, V_9, V'_{10}, V_{11}, V_{12}, V_{13}, V'_{14})\times \\
& P(V_5|V_1, V_3, V_4, V_6, V_7, V_8, V_9, V'_{10}, V_{11}, V_{12}, V_{13}, V'_{14})\times \\
& P(V'_{14}|V_1, V_3, V_4, V_6, V_7, V_8, V'_{10}, V_{11}, V_{12}, V_{13})\times \\
& P(V_3, V_{13}|V_6, V_7, V'_{10}, V_{12}, V_{13})P(V'_{10}|V_7, V_{11}, V_{12})P(V_{11}, V_{12}) \qquad (7)
\end{aligned}
$$

An estimand often corresponds to inference over a Bayesian network
Which is sometime very dense.

The treewidth of the above example is sqrt of n, when n is the number of variables

So, is evalution Exp(w)?.

Outline: Causal Inference

- Causal Models: Semantics
- Causal Models: Queries
- Identifiability
- **Estimand Methods**
- Learning Methods

The Plug-in estimate

- The Plug-in methods uses the “empirical distributions extracted from the data to estimate observed probabilistic quantities.
- Complexity of generating a table is O(|D|).
- Complexity of evaluation is exponential in the hyper-tree width.
- Computation can explore the graph and sparseness of the probabilistic quantities.

Empirical Factors, Sparse Representation

Table 80: 6 Variables with domain size 3

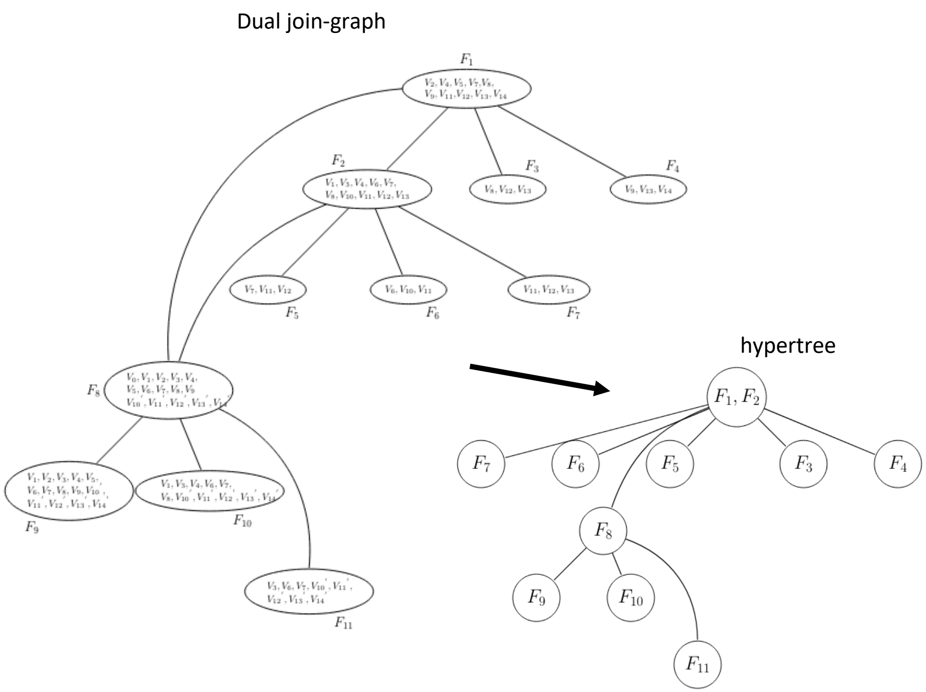
(a) Data Table

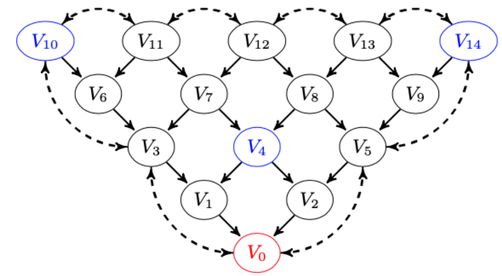
A	B	C	D	E	F
0	1	2	0	1	2
1	2	0	1	2	0
2	0	1	2	0	1
1	0	2	1	0	2
2	1	0	2	1	0
0	1	2	0	1	2
1	2	0	1	2	0
2	0	1	2	0	1
1	0	2	1	0	2
0	1	2	0	1	2
1	2	0	1	2	0
2	0	1	2	0	1
1	0	2	1	0	2
2	1	0	2	1	0
0	0	0	0	0	0
1	2	1	0	2	0
0	1	0	2	1	1
0	0	0	0	0	0
2	0	1	2	0	1
0	2	2	1	0	1

(b) Sparse Factor Table

A	B	C	D	E	F	Probability
0	1	2	0	1	2	0.20
1	2	0	1	2	0	0.20
2	0	1	2	0	1	0.20
1	0	2	1	0	2	0.15
2	1	0	2	1	0	0.10
0	0	0	0	0	0	0.10
1	2	1	0	2	0	0.05
0	1	0	2	1	1	0.05
0	2	2	1	0	1	0.05

Estimands Tree-Decomposition (Cones)





(b) Cone Cloud, $n = 15$ (15-CC)

$$
\begin{aligned}
P(V_0|V_{14}, V_{10}, V_4) = &\\
\sum_{V_1,V_2,V_3,V_5,V_6,V_7,V_8,V_9,V_{11},V_{12},V_{13},v'_{10},V'_{11},V'_{12},V'_{13},V'_{14}} & P(V_2|V_4, V_5, V_7, V_8, V_9, V_{11}, V_{12}, V_{13}, V_{14})\times \\
& P(V_9|V_{13}, V_{14})P(V_8|V_{12}, V_{13})P(V_1|V_3, V_4, V_6, V_7, V_8, V_{10}, V_{11}, V_{12}, V_{13})\times \\
& P(V_7|V_{11}, V_{12})P(V_6|V_{10}, V_{11})P(V_{11}, V_{12}, V_{13})\times \\
& P(V_0|V_1, V_2, V_3, V_4, V_5, V_6, V_7, V_8, V_9, V'_{10}, V_{11}, V_{12}, V_{13}, V'_{14})\times \\
& P(V_5|V_1, V_3, V_4, V_6, V_7, V_8, V_9, V'_{10}, V_{11}, V_{12}, V_{13}, V'_{14})\times \\
& P(V'_{14}|V_1, V_3, V_4, V_6, V_7, V_8, V'_{10}, V_{11}, V_{12}, V_{13})\times \\
& P(V_3, V_{13}|V_6, V_7, V'_{10}, V_{12}, V_{13})P(V'_{10}|V_7, V_{11}, V_{12})P(V_{11}, V_{12})
\end{aligned}
\tag{7}
$$

hw=2,w=14

Hyper-tree width: is the mximum number of functions placed in any cluster of a tree-decomposition

Complexity of Plug-In Scheme

Theorem: The complexity of evaluating an extimands whose expression has a tree-decomposition having hyper tree-width hw is O(n t^{hw}) if it has no denominators, where n = number of variables, t is the data size and k is the variables domain size. The complexity is also exponential in the tree-width is O(n k^{w}).

In all the examples we saw hw=1,2. w=1 or O($\sqrt{n}$).

But what about statistical accuracy?

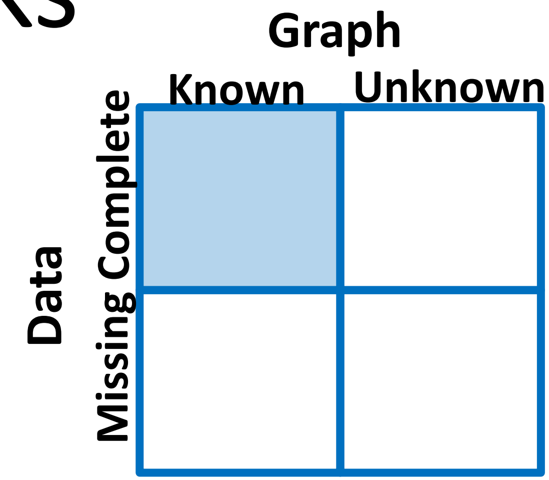
Note: The Plug-in can be viewed as using *maximum –likelihood* learning when data is fully observed over a Bayesian network graph extracted from the estimand.

Outline: Causal Inference

- Causal Models: Semantics
- Causal Models: Queries
- Identifiability
- Estimand Methods
- **Learning Methods**

Learning Bayesian networks

- Maximum Likelihood estimation
  - Select model that makes the data most probable
- For discrete X_i & no shared parameters
  - ML estimates are empirical probabilities

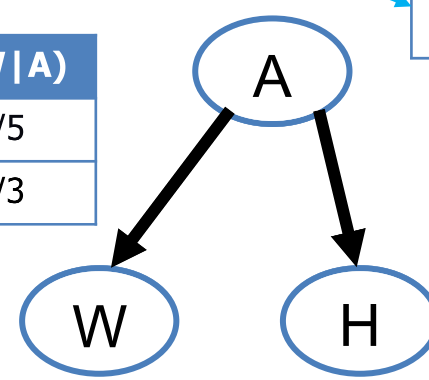


Observed Data

A	W	H
0	0	0
0	0	0
0	1	0
0	1	1
0	0	0
1	0	1
1	1	0
1	1	1

P(A)
3/8

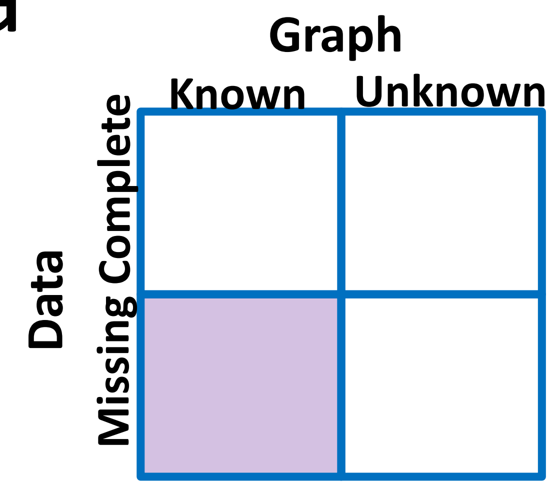
A	P(W|A)
0	2/5
1	2/3



A	P(H|A)
0	1/5
1	2/3

Learning with missing data

- Latent / hidden variables
  - Value is never observed
  - No unique model (e.g., symmetry)
  - No closed form solution; iterative ML
- More general missing values?
  - May depend on the reason for missingness!

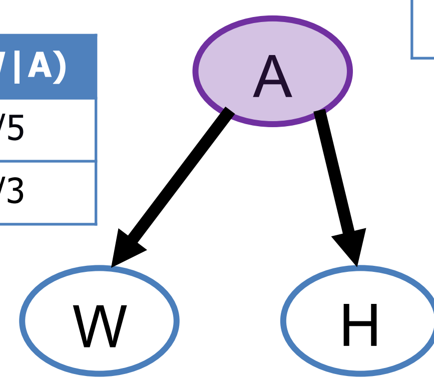


Observed Data

A	W	H
?	0	0
?	0	0
?	1	0
?	1	1
?	0	0
?	0	1
?	1	0
?	1	1

P(A)
3/8

A	P(W|A)
0	2/5
1	2/3



A	P(H|A)
0	1/5
1	2/3

Learning-Based Approach

Motivation: Use PGM algorithms for Causal Reasoning

$$P(V_7 \mid do(V_1)) = \sum_{V_2,V_3,V_4,V_5,V_6} P(V_6 \mid V_1,V_2,V_3,V_4,V_5) P(V_4 \mid V_1,V_2,V_3) P(V_2 \mid V_1)$$
$$\times \sum_{V_1'} P(V_7 \mid V_1',V_2,V_3,V_4,V_5,V_6) P(V_5 \mid V_1',V_2,V_3,V_4) P(V_3 \mid V_1',V_2) P(V_1')$$

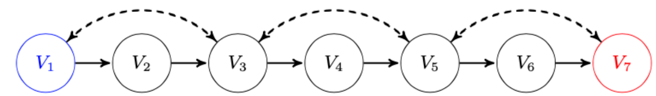


Figure 1: Chain Model with 7 observable variables and 3 latent variables

Motivation: Bucket-elimination on this network has tree-width 3 while the observational distribution has a tree-width of 7 and hypertree width of 1

Learning Idea:
Input: G, P(V)
1. Learn a full causal BN, from G and samples from P(V) using **EM**.
2. Truncate the learned model B into B_x.
3. Compute $P(\boldsymbol{Y})$ by Bucket elimination over B_x and return.
End Algorithm

Accuracy: EM is a mximum likelihood learning scheme that converges to a local maxima.
Complexity of Inference of both learning and inference is exponential in the **tree-width.**

Theorem: Given a model M yielding observational distribution P(V) and graph G, then any causal Bayesian Network over G having the same P(V), will agree with M **on any identifiable** causal effect query P(Y|do(X)).

Learning-Based Approach

What about the latent variables? Their domains?

Fit a good domain size for latent variables using the BIC score.

$$BIC_{\mathcal{B},\mathcal{D}} = -2 \cdot LL_{\mathcal{B},\mathcal{D}} + p \cdot \log(|\mathcal{D}|)$$

Algorithm 3: EM4CI

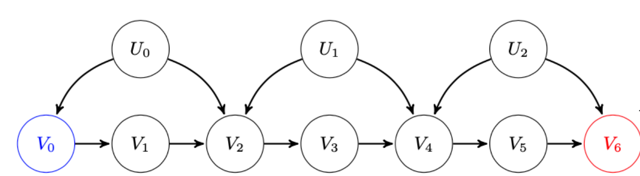
input : A causal diagram $\mathcal{G} = \langle \boldsymbol{U} \cup \boldsymbol{V}, E \rangle$, $\boldsymbol{U}$ latent and $\boldsymbol{V}$ observables; $\mathcal{D}$ samples from $P(\boldsymbol{V})$;
Query $Q = P(\mathbf{Y} \mid do(\mathbf{X} = \mathbf{x}))$.

output: Estimated $P(\mathbf{Y} \mid do(\mathbf{X} = \mathbf{x}))$

// k= latent domain size, $BIC_{\mathcal{B}} = BIC$ score of $\mathcal{B}$, $\mathcal{D}$,
// $LL_{\mathcal{B}}$ is the log-likelihood of $\mathcal{B}$, $\mathcal{D}$
Step 1: Initialize: $BIC_{\mathcal{B}} \leftarrow \inf$,
If $\neg$identifiable$(\mathcal{G}, Q)$, terminate.
For $k = 2, ...,$ to upper bound, **do**
  $(LL_{\mathcal{B}_{new}}) \leftarrow \max_{LL}\{\{\text{EM}(\mathcal{G}, \mathcal{D}, k) \mid i = 1, 2, \ldots, 10\}\}$;
  Calculate $BIC_{\mathcal{B}_{new}}$ from $LL_{\mathcal{B}_{new}}$
  If $BIC_{\mathcal{B}_{new}} \leq BIC_{\mathcal{B}}$,
    $\mathcal{B} \leftarrow \mathcal{B}_{new}$, $BIC_{\mathcal{B}} \leftarrow BIC_{\mathcal{B}_{new}}$
  else, break.
Endfor
$\mathcal{B}_{\mathbf{X}=\mathbf{x}} \leftarrow$ generate truncated CBN from $\mathcal{B}$.
return $\leftarrow$ evaluate $P_{\mathcal{B}_{\mathbf{X}=\mathbf{x}}}(\mathbf{Y})$

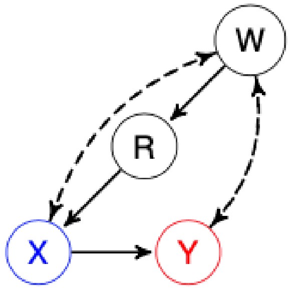
latent domains, CPT's parameters

U= {1,2,3}



P(V6|V5,U2)

Empirical Evaluation: Small Graphs

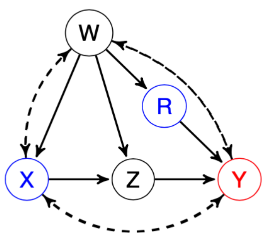


(a) Model 1

Small Synthetic models

Estimand Expressions for Models 1 & 8 .

Model	Estimate of $P(Y \mid do(X))$
1	$\frac{\sum_W P(X,Y\mid R,W)P(W)}{\sum_W P(X\mid R,W)P(W)}$
8	$\sum_{R,W,Z} P(Z\mid R,W,X)P(R\mid W)\sum_x P(Y\mid R,W,x,Z)P(x\mid R,W)P(W)$



(c) Model 8

Results for EM4CI and Plug-in estimates on $P(Y=y|do(X=0))$, $(d,k)=(2,10)$, k_{lrn} is the learned domain sizes of latent variables.

	100 Samples				1,000 Samples			
	EM4CI			Plugin	EM4CI			Plugin
Model	(LL,BIC)	(mad,time(s))	k_{lrn}	(mad,time)	(LL,BIC)	(mad,time(s))	k_{lrn}	(mad,time)
1	(-79,167)	(0.00348, 0.13)	2	(0.015, 0.016)	(-716,1445)	(0.003475, 1.5)	2	(0.002, 0.21)
8	(-121,262)	(0.0373, 0.61)	2	(0.288 , 0.016)	(-1290, 2609)	(0.0083, 6.6)	2	(0.154, 0.248)

Empirical Evaluation: Large Synthetic

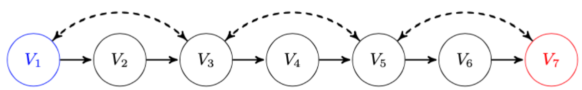
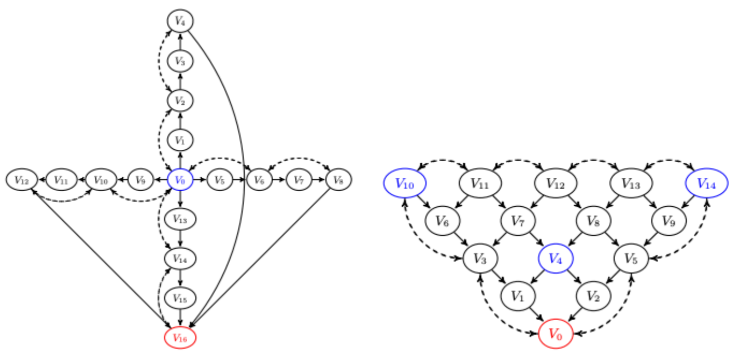


Figure 1: Chain Model with 7 observable variables and 3 latent variables



(a) Diamond, $n = 17$ **(b)** Cone Cloud, $n = 17$

More accuracy for learning in all cases but 1

Table 78: Results for EM4CI and Plug-In on $P(Y|do(\mathbf{X}))$ $(d, k) = (4, 10)$

(a) 1,000 samples

Model	Query	k_{lrn}	EM4CI mad	EM4CI Learn-time(s)	EM4CI inf-time(s)	Plug-In mad	Plug-In time(s)
5-CH	$P(V_4\|do(V_0))$	4	0.0902	3.5	0.0001	0.1509	2.3
9-CH	$P(V_8\|do(V_0))$	4	0.1204	11.5	0.0002	0.1516	2.4
25-CH	$P(V_24\|do(V_0))$	2	0.0070	77.7	0.0003	0.0959	6.1
49-CH	$P(V_48\|do(V_0))$	4	0.0005	161.2	0.0007	0.0319	17.8
99-CH	$P(V_98\|do(V_0))$	6	0.0093	413.4	0.0023	0.0611	88.1
9-D	$P(V_8\|do(V_0))$	2	0.0719	24.6	0.0002	0.1832	3.4
17-D	$P(V_{16}\|do(V_0))$	6	0.0542	202.3	0.0006	0.0700	4.5
65-D	$P(V_{64}\|do(V_0))$	4	0.0074	432.4	0.0012	0.1716	232.5
6-CC	$P(V_0\|do(V_5))$	4	0.0088	23.5	0.0001	0.0156	2.3
15-CC	$P(V_0\|do(V_{14}))$	4	0.0147	60.8	0.0001	0.0659	4.5
45-CC	$P(V_0\|do(V_{14}, V_{36}, V_{44}))$	6	0.0097	199.2	2.7429	0.1509	18.6

(b) 10,000 samples

Model	Query	k_{lrn}	EM4CI mad	EM4CI Learn-time(s)	EM4CI inf-time(s)	Plug-In mad	Plug-In time(s)
5-CH	$P(V_4\|do(V_0))$	4	0.0508	17.3	0.0001	0.0537	2.5
9-CH	$P(V_8\|do(V_0))$	4	0.0236	150.0	0.0002	0.1074	3.1
25-CH	$P(V_24\|do(V_0))$	6	0.0068	697.1	0.0005	0.0714	26.4
49-CH	$P(V_48\|do(V_0))$	10	0.0017	2412.6	0.0036	0.0160	133.7
99-CH	$P(V_98\|do(V_0))$	6	0.0028	3887.9	0.0022	0.0433	850.6
9-D	$P(V_8\|do(V_0))$	4	0.0611	390.7	0.0002	0.1481	3.0
17-D	$P(V_{16}\|do(V_0))$	6	0.0360	1849.6	0.0007	0.0582	8.4
65-D	$P(V_{64}\|do(V_0))$	4	0.0022	4787.2	0.0013	0.1376	2258.5
6-CC	$P(V_0\|do(V_5))$	6	0.0138	116.9	0.0003	0.0136	2.7
15-CC	$P(V_0\|do(V_{14}))$	4	0.0022	489.5	0.0043	0.0321	10.9
45-CC	$P(V_0\|do(V_{14}, V_{36}, V_{44}))$	6	0.0026	1833.7	2.757	0.1561	105.8

Dependence on Model Size

Trajectory over Cone Instances **Trajectory over Chain Instances** **Trajectory over Diamond Instances**

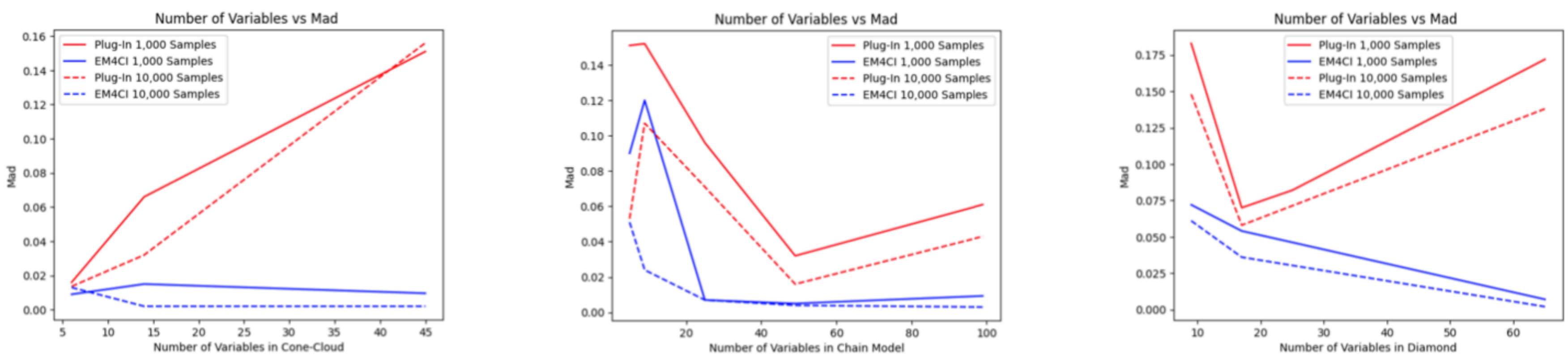


Figure 4: Comparing the accuracy of EM4CI and Plug-In. While both methods improve with more samples (solid to dashed lines), the error (mad) of EM4CI is smaller, even when compared to Plug-In with more samples.

Empirical Evaluation: Results

- **4 Real Networks**
- “Alarm”, “A”, “Barley”, and “Win95pts” network

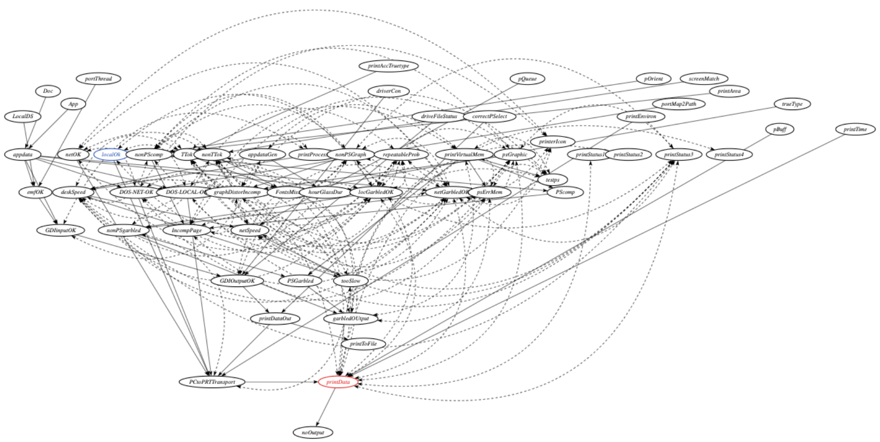


Figure: A Model

Table 8: Plug-In & EM4CI results on the **A Network**
$|\boldsymbol{V}| = 46;\ |\boldsymbol{U}| = 8;\ d = 2;\ k = 2$ treewidth ≈ 16

(a) Plug-In

	1,000 Samples		*10,000 Samples*	
Query	mad	time(s)	mad	time(s)
$P(V_{51}\|\text{do}(V_{10}))$	0.0584	8.0	0.0114	55.7
$P(V_{51}\|\text{do}(V_{14}))$	0.0319	8.3	0.0056	51.3
$P(V_{51}\|\text{do}(V_{41}))$	0.0255	13.9	0.0092	48.3
$P(V_{51}\|\text{do}(V_{45}))$	0.0496	9.8	0.0206	49.1

(b) EM4CI

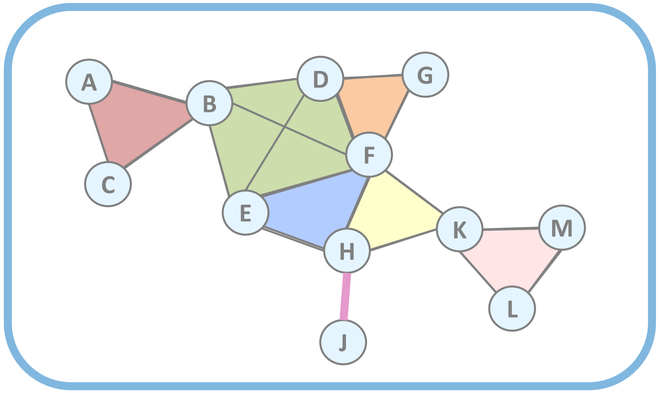
	1,000 Samples		*10,000 Samples*	
Learning	$time = 71(s)$	$k_{lrn} = 4$	$time(s) = 541$	$k_{lrn} = 4$
Inference: Query	mad	time(s)	mad	time(s)
$P(V_{51}\|\text{do}(V_{10}))$	0.0139	0.0012	0.0083	0.0012
$P(V_{51}\|\text{do}(V_{14}))$	0.0143	0.0047	0.0086	0.0046
$P(V_{51}\|\text{do}(V_{41}))$	0.0147	0.0042	0.0079	0.0041
$P(V_{51}\|\text{do}(V_{45}))$	0.0140	0.0031	0.0082	0.0030

Summary: Causal queries

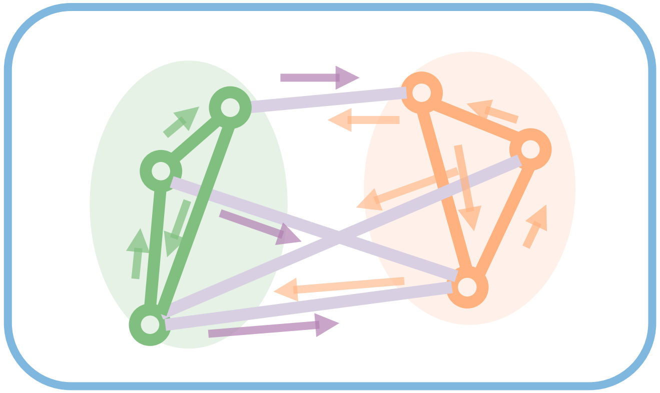
- Causal Bayesian networks and SCM encodes causal assumptions explicitly.
- Causal effects and counterfactual queries can be computed from the full causal model by PGM methods.
- Given only the causal diagram and observational data queries can be evaluated if identifiable.
- Causal effect queries can be done by statistical estimation of estimands (defined by observation quantities).
- Estimands can be generated by Backdoor, Frontdoor, do-calculous. The ID algorithm.
- Model completion by learning is a promising alternative for causal inference that exploit PGM methods.

Summary of Lectures

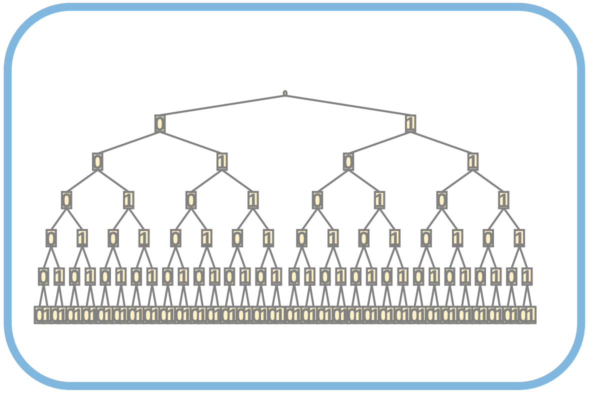
Class 1: Introduction & Inference



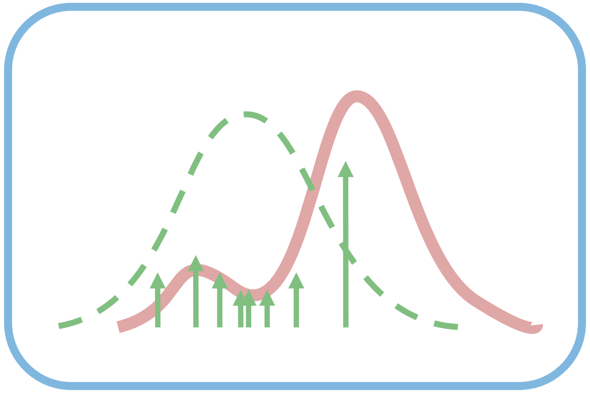
Class 2: Bounds & Variational Methods



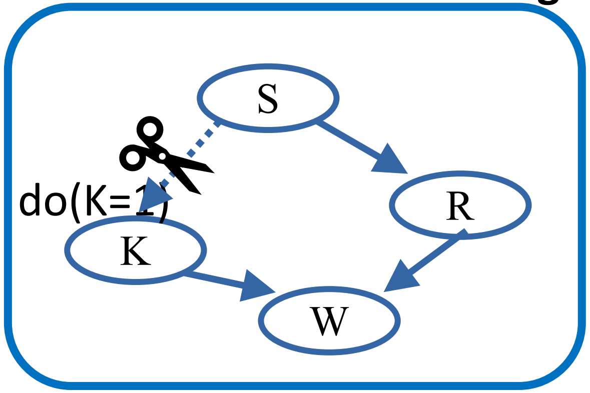
Class 3: Search Methods



Class 4: Monte Carlo Methods



Class 5: Causal Reasoning



Software

- My software page

pyGMs : Python Toolbox for Graphical Models by Alexander Ihler.

UAI Probabilistic Inference Competitions

- **2006**

(aolib)

- **2008**

(aolib)

- **2012**

(daoopt)

- **2014**

(daoopt)

(daoopt)

(merlin)

Marginal Map

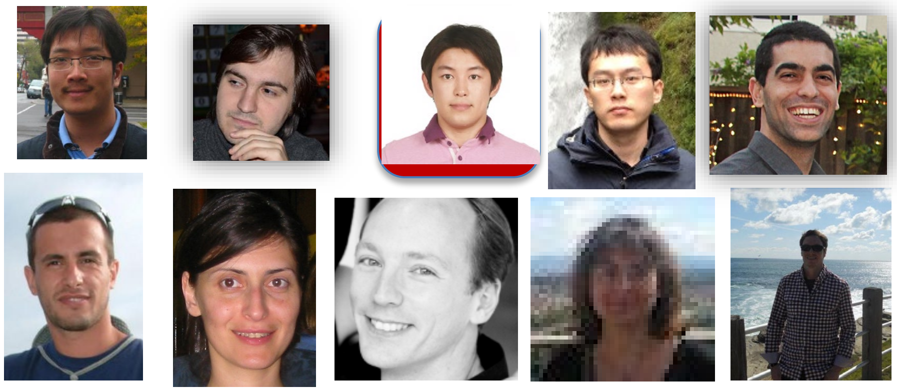
New UAI Competition

- [UAI Competition 2022]

Solver	20sec	1200sec	3600sec
uai14-pr	61.7	96.8	96.7
ibia-pr	53.6	96.6	97.1
AbstractionSampling	78.9	91.7	93.9
lbp-pr	90.3	89.9	90.2

Thank You !

For publication see:
http://www.ics.uci.edu/~dechter/publications.html



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