Formal Aspects of Strategic Reasoning and Game Playing

Laurent Perrussel – Munyque Mittelmann – Nello Murano

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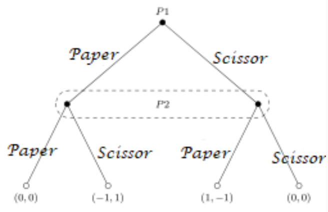
IRIT – Université Toulouse Capitole (FR)
University of Napoli (IT)

Motivation

Motivation(1/2)

- Game: describe and justify actions in a multi-agent context



- Autonomy for agent means
  - Decision making: justify actions (agent rationality)
    P1 plays scissor because...
  - handling or playing in different environments (facing a new game)
    P2 now plays Tic-tac-toe

Motivation (2/2)

Computer Science vs Game Theory?

- Game Theory
  Main goal: assessing the graph (i.e. the game) and find equilibrium or existence of winning strategies
- Computer Science
  Main goal: compact representation, computation of the possible next actions and choice

General Game Playing
Computer scientists challenge: build programs sufficiently general for playing different games.

Week Organization

- **Lecture 1**: Game Description Language and Game Description Logic (GDL)
- **Lecture 2**: GDL and Imperfect Information
- **Lecture 3**: Basics of Formal Verification of 1 and 2 players Game
- **Lecture 4**: Strategic Reasoning and Formal Verification of multiple players Game
- **Lecture 5**: Strategic Reasoning and Quantitative information and goals

Game Description - Organization

Motivation

General Game Playing

Game Description Language

Game Description Logic: GDL with a (logic-flavored) semantics

Imperfect Information: Extending the Logic

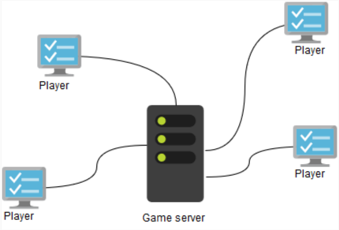
Reasoning for winning?

Still a lot to do! - Example: Equivalent games

Perspectives

General Game Playing

General Game Playing - Overall organization



More details at `http://ggp.org` and in [Genesereth and Thielscher, 2014]

Interaction between server and players:

$\Rightarrow$ Game rules & current state of the game

$\Leftarrow$ Moves

General Game Playing - Prerequisites

Limited to the shared aspect of the game

- **Type of game** No randomness - perfect information (board game)
- **Language** Processable by the server and players (game rules)
- **Timeclock** sync player moves and game run

  No prerequisite on players implementation (reasoning is not compulsory!)

General Game Playing - Key challenge

Overall goal: designing intelligent agent

Building players sufficiently general for playing different games

GGP competition: players compete by playing at different games.

Challenge is ***not*** *to build the best player for one game*

GGP player will never beat AlphaGo (at least in a Go game!)

General Game Playing - Specialized player

- Usually rules of the game hard-coded in the player
- Possibly exhaustive search
- Predefined library of best moves (tactics, ie. library of plans) combined with heuristics
- Library can be learned

General Game Playing - Representing games

Game Description Language (GDL)

- General
  General enough for describing different games: no primitives related to some specific game
- Game rules and remarkable states
  Initial and final states, legal actions...
- Compact
  *Logic-based language, namely **first-order logic***

General Game Playing - Processing GDL

Server

- Not relevant - Zero intelligence

Players

- No specific implementation
  Several implementation are available (Java, Prolog...)
- No specific way to play
  Reasoning, Heuristics, Monte-Carlo, CSP...

General Game Playing - GDL Example

Tic-Tac-Toe

Tic Tac Toe (or Noughts and Crosses, Xs and Os) is a game for two players who take turns placing their marks in a 3x3 grid. The first player to place three of his marks in a horizontal, vertical, or diagonal row wins the game.

General Game Playing - GDL Example

Tic-Tac-Toe GDL representation (1/3)

```
;;; Components
;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;
    (role white)
    (role black)
    ...

;;; init
;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;
    (init (cell 1 1 b))
    ...
    (init (cell 3 3 b))
    (init (control white))
```

General Game Playing - GDL Example

Tic-Tac-Toe GDL representation (2/3)

```
;;; legal moves
    (<= (legal ?w (mark ?x ?y))
        (true (cell ?x ?y b))
        (true (control ?w)))

    (<= (legal white noop)
        (true (control black)))
    ...

;;; next (effects)
    (<= (next (cell ?m ?n x))
        (does white (mark ?m ?n))
        (true (cell ?m ?n b)))
    ...
```

General Game Playing - GDL Example

Tic-Tac-Toe GDL representation (3/3)

```
;;; goal
    (<= (goal white 100)
        (line x)
        (not (line o)))

    (<= (goal white 0)
        (not (line x))
        (line o))
    ...

;;; terminal

    (<= terminal
        (line x))
    ...
```

Game Description Language

GDL - Primitives

Prolog/Datalog like rules with predefined keywords (prefix notation)

Static perspective

- `role` players of the game
  (role white)
- `init` initial state
  (init (cell 1 1 b))
- `terminal` terminal state
  (<= terminal (line x))
- `true` current state
  (true (cell 2 2 b))

GDL - Primitives (2/2)

Dynamic perspective

- `legal` rules of the game - possible moves
  (<= (legal x noop) (true (control o)))
- `does` performing action (in the current state)
  (<= (next (cell ?x ?y ?player)) (does ?player (mark ?x ?y)))
- `next` update function
  (<= (next (control o)) (true (control x)))
- `goal` objectives of the players
  (<= (goal ?player 100) (line ?player))

GDL - syntax constraints

"Enforcing" game flavor

- sequence of keywords is *prohibited*
- `role` only atomic (fixed players)
- `next` predicate only in heads
- `init` and `true` predicates only in bodies
- `does` predicate only in bodies
- recursion restriction

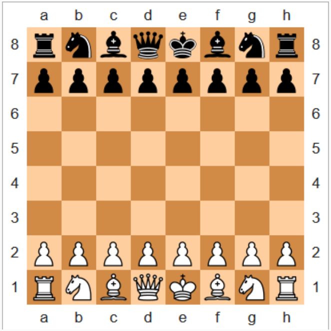
GDL - semantics

A logic programming perspective

- Minimal data set D which are models of a game G: set of grounded atoms
    - ground literal (`not p`) is satisfied iff `p` is not in D
- GDL game description: logic program with predefined predicate and shape
    - Complete definition of `role`, `init`
    - `legal` and `goal` only defined wrt `true`
    - `next` only defined wrt `true` and `does`
- Unique minimal model satisfying the state of the game (ie `true` predicate)
- Several minimal models when considering the dynamics (ie `does` predicate)

GDL - Chess example (1/6)

- Around 1000 lines!
- initial state already complex
- legal moves differ for each piece type
- basic rules + specific rules (pawn promotion...)
- no number in GDL: rules for encoding them!



GDL - Chess example (2/6)

Initial state

- Two players
- Chess board and pieces
    - blank cells
    - black and white rooks (wr, br)
    - black and white pawn (wp)
- First player

```
(role white)
(role black)
(init (cell a 1 wr))
(init (cell a 2 wp))
(init (cell a 3 b))
...
(init (cell h 8 br))
(init (control white))
```

GDL - Chess example (3/6)

Goal states

- Check mate the opponent
  - ⇒ should be defined for the `white` and `black` players
- Draw is a good compromise
- Not being checkmate is also a goal!

```
...
(<= (goal white 100)
    (checkmate black))
(<= (goal white 50)
    stalemate)
(<= (goal white 0)
(checkmate white))
...
```

GDL - Chess example (4/6)

End of the game

- One player is stuck
  - ⇒ regardless king is in check or not
- After 200 rounds, game is stopped
  - ⇒ Numbers and counting should be defined

```
(<= (stuck ?pl)
    (role ?pl)
    (not (has_legal_move ?pl)))
...
(<= terminal
    (true (control ?player))
    (stuck ?player))
(<= terminal
    (true (step 201)))
...
(succ 1 2)
(succ 2 3)
...
```

GDL - Chess example (5/6)

Legal moves

- Define the moves for each piece
  - what means adjacent?
  - what means diagonal?
  - ...
- Define legality
  - context is OK (players, piece is on the cell, move is meaningful...)

```
(<= (knight_move ?piece ?u ?v
                 ?x ?y ?owner)
    (piece_owner_type ?piece
                ?owner knight)
    (adjacent_two ?v ?y)
    (adjacent ?u ?x))
...
(<= (legal ?player (move ?piece
                     ?u ?v ?x ?y))
  (true (control ?player))
  (true (cell ?u ?v ?piece))
  (occupied_by_opp ?x ?y ?player)
  (legal2 ?player (move ?piece
                   ?u ?v
                   ?x ?y))
   ...
```

GDL - Chess example (6/6)

Actions and update

- General rules for the game

  e.g. blank cell
- specific rules for specific moves

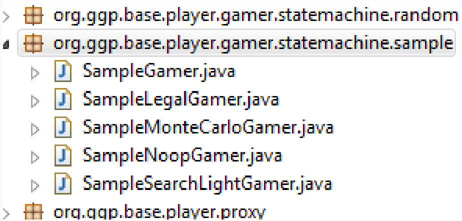
  e.g. "en passant"
- update the step number

```
(<= (next (cell ?u ?v b))
      (does ?player (move ?p ?u
                ?v ?x ?y)))

(<= (next (cell ?x1 ?y1 b))
    (does ?player (move ?piece
       ?x1 ?y1 ?x2 ?y2))
     (pawn_capture_en_passant
       ?player ?x1 ?y1 ?x2 ?y2))

(<= (next (step ?y))
    (true (step ?x))
        (succ ?x ?y))
```

Implementing a Player

- Free implementation
- Reasoning is not compulsory
- Main technique:
  - Search-Space and Heuristics
  - Compute the value of the next state

eg. (1) Minimax
eg. (2) Monte-Carlo Tree Search



Game Description Logic: GDL with a (logic-flavored) semantics

Signature and Language

Towards reasoning about Perfect Information Games

First step is to build a logic based on GDL [Jiang, 2016]

Signature Agents, actions, propositions:

$$(N, \mathcal{A}, \Phi)$$

Language predefined symbols and temporal operators

$$\varphi ::= p \mid \mathit{initial} \mid \mathit{terminal} \mid \mathit{legal}(r, a) \mid \mathit{wins}(r) \mid \mathit{does}(r, a) \mid \neg\varphi \mid \varphi \wedge \psi \mid \bigcirc\varphi$$

Tic-Tac-Toe

GDL description of Tic-tac-Toe:

1. $\mathit{initial} \leftrightarrow \mathit{turn}(\mathrm{x}) \wedge \neg \mathit{turn}(\mathrm{o}) \wedge \bigwedge_{i,j=1}^{3} \neg(p^{\mathrm{x}}_{i,j} \vee p^{\mathrm{o}}_{i,j})$
2. $\mathit{wins}(r) \leftrightarrow \bigvee_{i=1}^{3} \bigwedge_{l=0}^{2} p^{r}_{i,1+l} \vee \bigvee_{j=1}^{3} \bigwedge_{l=0}^{2} p^{r}_{1+l,j} \vee \bigwedge_{l=0}^{2} p^{r}_{1+l,1+l} \vee \bigwedge_{l=0}^{2} p^{r}_{1+l,3-l}$
3. $\mathit{terminal} \leftrightarrow \mathit{wins}(\mathrm{x}) \vee \mathit{wins}(\mathrm{o}) \vee \bigwedge_{i,j=1}^{3} (p^{\mathrm{x}}_{i,j} \vee p^{\mathrm{o}}_{i,j})$
4. $\mathit{legal}(r, a_{i,j}) \leftrightarrow \neg(p^{\mathrm{x}}_{i,j} \vee p^{\mathrm{o}}_{i,j}) \wedge \mathit{turn}(r) \wedge \neg \mathit{terminal}$
5. $\mathit{legal}(r, \mathit{noop}) \leftrightarrow \mathit{turn}(-r)$
6. $\bigcirc p^{r}_{i,j} \leftrightarrow p^{r}_{i,j} \vee (\mathit{does}(r, a_{i,j}) \wedge \neg(p^{\mathrm{x}}_{i,j} \vee p^{\mathrm{o}}_{i,j}))$
7. $\mathit{turn}(r) \rightarrow \bigcirc \neg \mathit{turn}(r) \wedge \bigcirc \mathit{turn}(-r)$

State-Transition Model (Perfect-Information Game)

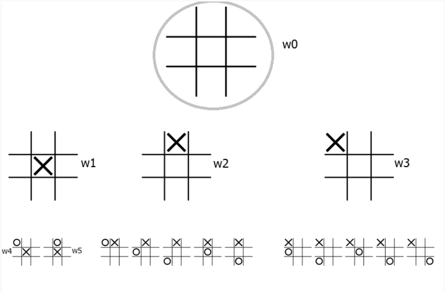
$$M = (W, I, T, L, U, g, \pi)$$

- W is a non-empty finite set of *possible states*.
- $I \subseteq W$, representing a set of *initial* states.
- $T \subseteq W \setminus I$, representing a set of *terminal* states.
- $L \subseteq W \backslash T \times N \times 2^{\mathcal{A}}$ is a *legality* relation, specifying legal actions for each agent at non-terminal states. Let $L_r(w) = \{a \in \mathcal{A} \ : \ (w, r, a) \in L\}$ be the set of all legal actions for agent r at state w. To make the game playable, we require $L_r(w) \neq \emptyset$ for every $r \in N$ and $w \in W \backslash T$.
- $U : W \times \mathcal{A}^{|N|} \rightarrow W \backslash I$ is an *update* function, specifying the state transition for each state and *joint action* (synchronous moves).
- $g : N \rightarrow 2^W$ is a *goal* function, specifying the winning states of each agent.
- $\pi : W \rightarrow 2^{\Phi}$ is a standard valuation function.

ST Model - Details (1/3)

$M = (W, I, T, L, U, g, \pi)$

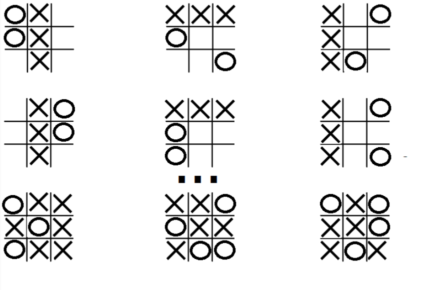
- Set of states W can be very large
  5 478 states for Tic-Tac-Toe
- Set $I = \{w_0\}$ usually a singleton



ST Model - Details (2/3)

$M = (W, I, T, L, U, g, \pi)$

- Set T of terminal states consider all cases
  - *958 terminal states*
- winning or draw states
- winning states g specific to each agent and subset of T



ST Model - Details (3/3)

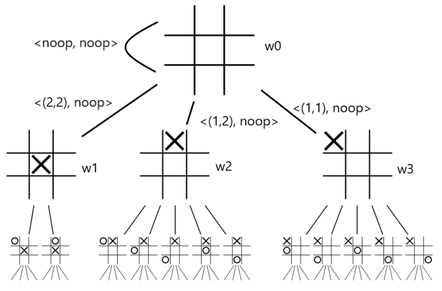
$M = (W, I, T, L, U, g, \pi)$

- Legal transitions (L)

  9 legal actions from $\langle(1,1), noop\rangle$ *to* $\langle(3,3), noop\rangle$ *in* w_0

- Update is deterministic.

  Update can be defined while illegal (eg. $\langle noop, noop\rangle$



Path

Path δ is an infinite sequence of states and actions

$$w_0 \xrightarrow{d_1} w_1 \xrightarrow{d_2} w_2 \cdots \xrightarrow{d_j} \cdots$$

such that for all $j \geq 1$ and for any $r \in N$,

1. $w_j = U(w_{j-1}, d_j)$ (state update);
2. $(w_{j-1}, d_j(r)) \in L_r$ (that is, any action that is taken must be legal);
3. if $w_{j-1} \in T$, then $w_{j-1} = w_j$ (that is, a loop after reaching a terminal state).

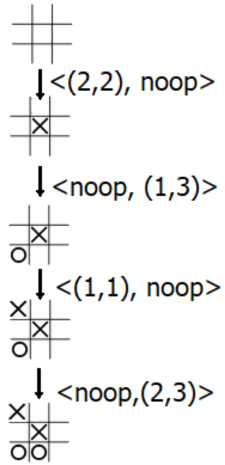
$\theta_r(\delta, j)$: action of agent r at stage j of δ

Path

Sequence of actions

- Run over an ST-model
- No requirement about first and last states
- formulas will be interpreted over a path at some step
- $\delta[j]$: j*th* state of path δ
- $\theta_r(\delta, j)$ action performed by agent r at state j of path δ

$$\text{eg: } \theta_x(\delta, 3) = a_{1,1}$$



Semantics

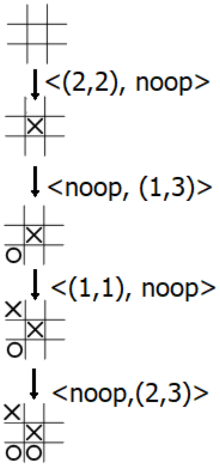
W.r.t. M, some path δ and index j

$M, \delta, j \models p$	iff	$p \in \pi(\delta[j])$
$M, \delta, j \models \neg\varphi$	iff	$M, \delta, j \not\models \varphi$
$M, \delta, j \models \varphi_1 \wedge \varphi_2$	iff	$M, \delta, j \models \varphi_1$ and $M, \delta, j \models \varphi_2$
$M, \delta, j \models \mathit{initial}$	iff	$\delta[j] \in I$
$M, \delta, j \models \mathit{terminal}$	iff	$\delta[j] \in T$
$M, \delta, j \models \mathit{wins}(r)$	iff	$\delta[j] \in g(r)$
$M, \delta, j \models \mathit{legal}(r, a)$	iff	$a \in L_r(\delta[j])$
$M, \delta, j \models \mathit{does}(r, a)$	iff	$\theta_r(\delta, j) = a$
$M, \delta, j \models \bigcirc\varphi$	iff	$M, \delta, j+1 \models \varphi$

Path

Tic-Tac-Toe formulas

- $M, \delta, 0 \models \neg p^x_{1,1}$
- $M, \delta, 1 \models p^x_{2,2}$
- $M, \delta, 1 \models \neg wins(x)$
- $M, \delta, 1 \models does(o, a_{1,3})$
- $M, \delta, 2 \models \bigcirc does(o, a_{2,3})$
- $M, \delta, 3 \models \neg \bigcirc wins(x)$



GDL for reasoning about games

General game properties

- $\models \bigvee_{r \in N} wins(r) \rightarrow terminal$ iff $g(r) \subseteq T$

Bounded time

- $\not\models \bigwedge_{i \in 1..n} \bigcirc^{i} \neg wins(r) \rightarrow \bigcirc^{n+1} \neg wins(r)$

[0] $\bigcirc^{n}$: sequence of n $\bigcirc$

GDL for reasoning about games

General game playing w.r.t. some ongoing game

- assessing a "strategy" vs ⟨game state, move⟩ (M, δ)
- Look ahead via model checking ($\mathbf{PTIME}$)
- Winning move (encoded in δ)?

$$M, \delta, 0 \models \bigcirc wins(x)$$

- Prevent opponent x to win?
    - Choose an action a for x and an action b for $-x$ next move

    ⇒ Check $M, \delta, 0 \models \bigcirc does(-x, b) \wedge \bigcirc^2 wins(-x)$
    - Choose alternative action a' for x

    ⇒ Check $M, \delta', 0 \models \bigcirc does(-x, b) \wedge \bigcirc^2 \neg wins(-x)$
    - Choose other b' and recheck
- No meta-reasoning in GDL (assessment over paths)

  "Try to win, if not prevent to loose" cannot be represented

GDL for reasoning about games

Specific game properties

- Set of rules specific to a game
- Identify pattern for general game playing
- Example: Tic-Tac-Toe
  - $diagonal(x) \leftrightarrow \bigwedge_{i \in 1..3} p^x_{i,i} \vee \bigwedge_{i \in 0..2} p^x_{1+i,3-i}$
  - $line(x) \leftrightarrow diagonal(x) \vee column(x) \vee row(x)$
  - Double threat consequence of move a by x: two potential lines
  - Meta-reasoning as two paths are considered (eg: row or column):

    For any next move b by $-x$, pick up x move c and c', build path δ, δ' and check

$$M, \delta, 0 \models \bigcirc^2 row(x) \text{ or } M, \delta', 0 \models \bigcirc^2 column(x)$$

Exercise

(Simplified) Nim Game

- 2 players sequential game
- 12 sticks
- at each round, each player picks 1, 2 or 3 sticks
- winner of game: the player picking the last stick

Provide the GDL representation

Imperfect Information: Extending the Logic

Imperfect Information

Example

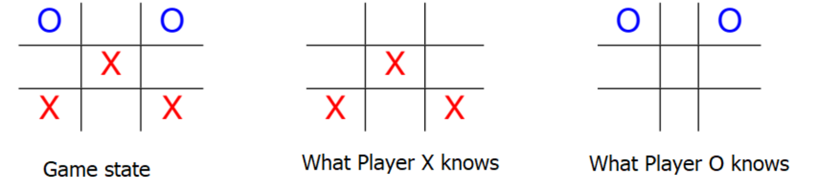


Figure 1: Krieg Tic-Tac-Toe

Two players black, white

- see her own marks only
- know turn-taking and available actions

Main issue

How to describe and reason about games with imperfect information?

Epistemic extension of GDL logic

Server side vs Player side

- Player perspective
  - How to handle certain and uncertain information?
  - How to handle other players' "knowledge"?
- Server Perspective
  - GDL-II: how to represent imperfect information?
  - GDL-II: how Information flows
  - GDL-II: randomness

Epistemic extension: Syntax (1/2)

Extending GDL with epistemic operators [Jiang et al., 2021]

- $\mathsf{K}_r\varphi$: "agent r knows φ"
- $\mathsf{C}\varphi$: as "φ is common knowledge among all the agents in N"

Definition (Syntax)

$$\varphi ::= p \mid initial \mid terminal \mid legal(r,a) \mid wins(r) \mid does(r,a) \mid$$
$$\neg\varphi \mid \varphi \wedge \psi \mid \bigcirc\varphi \mid \mathsf{K}_r\varphi \mid \mathsf{C}\varphi$$

$\mathsf{E}\varphi =_{def} \bigwedge_{r \in N} \mathsf{K}_r\varphi$

Epistemic extension: Syntax (2/2)

Sequential Krieg-Tic-Tac-Toe - Epistemic rules

Additional symbol:

$tried(r, a_{i,j})$ represents the fact that player r has tried to mark cell (i,j) but failed

1. $tried(r, a_{i,j}) \rightarrow p^{-r}_{i,j}$
2. $does(r, a_{i,j}) \rightarrow \mathsf{K}_r(does(r, a_{i,j}))$
3. $initial \rightarrow \mathsf{E}initial$
4. $(turn(r) \rightarrow \mathsf{E}turn(r)) \wedge (\neg turn(r) \rightarrow \mathsf{E}\neg turn(r))$
5. $(p^r_{i,j} \rightarrow \mathsf{K}_r p^r_{i,j}) \wedge (\neg p^r_{i,j} \rightarrow \mathsf{K}_r \neg p^r_{i,j})$
6. $(tried(r, a_{i,j}) \rightarrow \mathsf{K}_r tried(r, a_{i,j})) \wedge (\neg tried(r, a_{i,j}) \rightarrow \mathsf{K}_r \neg tried(r, a_{i,j}))$

Epistemic extension: Semantics (1/2)

Epistemic state transition (EST) model M is a tuple
$(W, I, T, \{R_r\}_{r\in N}, \{L_r\}_{r\in N}, U, g, \pi)$

- W is a non-empty set of *possible states*.
- $I \subseteq W$, representing a set of *initial* states.
- $T \subseteq W \backslash I$, representing a set of *terminal* states.
- $R_r \subseteq W \times W$ is an equivalence relation for agent r, indicating the states that are indistinguishable for r.
- $L_r \subseteq W \times A^r$ is a *legality* relation for agent r,
- $U : W \times \prod_{r\in N} A^r \hookrightarrow W\backslash I$ is a partial *update* function
- $g : N \to 2^W$ is a *goal* function, specifying the winning states for each agent.
- $\pi : W \to 2^\Phi$ is a standard valuation function.

Epistemic extension: Semantics (2/2)

Imperfect Recall

$$\delta \approx_r \delta' \quad \text{iff} \quad \delta[0]\, R_r\, \delta'[0]$$

Satisfaction with respect to some EST M and path δ

$$
\begin{array}{lll}
M, \delta \models \mathsf{K}_r\varphi & \text{iff} & \text{for any } \delta' \in \mathcal{P}\text{, if } \delta \approx_r \delta'\text{, then } M, \delta' \models \varphi \\
M, \delta \models \mathsf{C}\varphi & \text{iff} & \text{for any } \delta' \in \mathcal{P}\text{, if } \delta \approx_N \delta'\text{, then } M, \delta' \models \varphi
\end{array}
$$

where $\approx_N$ is the transitive closure of $\bigcup_{r \in N} \approx_r$ and $\mathcal{P}$ is the set of all paths in M.

EGDL for reasoning about games

General game playing w.r.t. some ongoing game

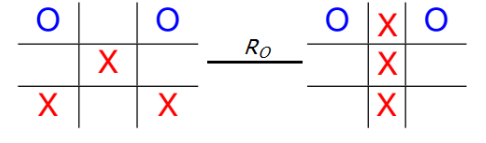


Figure 2: Player o Knowledge

Player o cannot distinguish between the two states

$terminal \rightarrow Cterminal$ is not valid

Epistemic extension: Basic properties

Properties about Krieg-Tic-Tac-Toe (valid formulas in all Krieg-Tic-Tac-Toe models)

1. $\mathit{initial} \rightarrow \mathsf{C}\mathit{initial}$
2. $\mathit{legal}(a^r_{i,j}) \rightarrow \mathsf{K}_r(\mathit{legal}(a^r_{i,j}))$
3. $\mathit{does}(a^r_{i,j}) \rightarrow \bigcirc \mathsf{K}_r(p^r_{i,j} \vee \mathit{tried}(a^r_{i,j}))$
4. $\mathsf{K}_r \mathit{tried}(a^r_{i,j}) \rightarrow \mathsf{K}_r p^{-r}_{i,j}$

EGDL for reasoning about games

General game playing w.r.t. some ongoing game

- assessing a "strategy" vs ⟨game state, move⟩ (M, δ)
- Looking ahead via model checking (Δ_2^p)
- Winning situation (encoded in δ)?

$$M, \delta \models K_r \bigcirc wins(x)$$

- Prevent opponent of r to win?

$$\text{Check } M, \delta \models does(r, a) \wedge K_r \bigcirc \neg wins(-r)$$

- Opponent of r may win (wrt. to some r move)?

$$\text{Check } M, \delta \models \neg K_r \neg \bigcirc (does(-r, a) \wedge \bigcirc wins(-r))$$

- No complex reasoning over paths in EGDL

EGDL for reasoning about games

Specific game properties

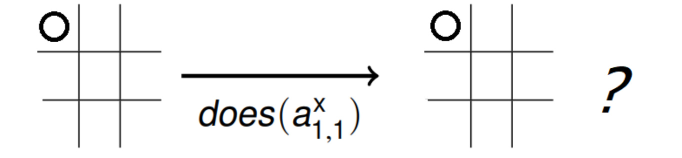


Figure 3: Player x move

Player x knows that

$$does(x, a_{i,j}) \rightarrow \bigcirc K_x(p^x_{i,j} \vee tried(x, a_{i,j}))$$

Hence

$$K_x tried(x, a_{1,1})$$

EGDL for reasoning about games

Specific game properties

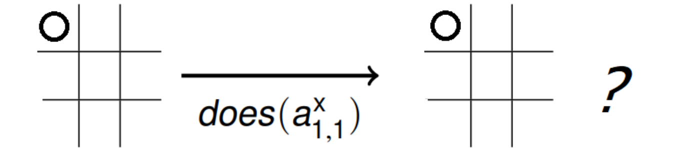


Figure 4: Player x move

Player x knows that

$$K_x tried(x, a_{i,j}) \rightarrow K_x p^o_{i,j}$$

Hence

$$K_x p^o_{1,1}$$

Exercise

Guessing a number

- 2 players game
- Player 1 choose a number $n \in [1, 10]$ (initial state)
- Player 2 has to guess n
- After each round, Player 1 informs Player 2 whether its proposal is too low or too high.
- Player 2 wins if it guesses n in 3 rounds.

Provide the EGDL representation

GDL-II - Representing Imperfect Information Games

GDL-II: extension of GDL - Server side [Thielscher, 2010]

- `sees` specify what a player perceives at the next state

  ```
  (sees ?player (holds ?player ?card))
  ```

  `sees` behaviour similar to `next`: only in head of clauses.

- *random* random player

  ```
  (role random)
  ```

  Perform action with parameters randomly set

  ```
  (does random (deal ?player ?card))
  ```

Krieg Tic-Tac-Toe GDL representation (1/3)

Simultaneous move: possible tie-break

```
;;; additional random player for tie break
    (role black)
    (role white)
    (role random)

;;; random player can only solve tie break
    (legal random (tiebreak white))
    (legal random (tiebreak black))

;;; "tried" predicate: "try to mark"
    (<= next (tried ?r ?m ?n)
        (does ?r (mark ?m ?n)))

   (<= next (tried ?r ?m ?n)
        (true (tried ?r ?m ?n)))
```

Krieg Tic-Tac-Toe GDL representation (2/3)

Solving tie-break (simultaneous moves)

```
;;; possible tie-break
    (<= next (cell ?m ?n ?r)
        (true (cell ?m ?n b))
        (does white (mark ?m ?n)))
        (does black (mark ?m ?n)))
        (does random (tiebreak ?r)))
```

Krieg Tic-Tac-Toe GDL representation (3/3)

Only seeing own moves - simultaneous moves

```
;;; success when moves differ
    (<= sees ?r1 (cell ?m1 ?n1 ?r1)
        (true (cell ?m1 ?n1 b))
        (does ?r1 (mark ?m1 ?n1))
        (does ?r2 (mark ?m2 ?n2))
        (distinct ?m1 ?m2))


 ...
;;; successful tie break
    (<= sees black (cell ?m ?n black)
        (true (cell ?m ?n b))
        (does black (mark ?m ?n)))
        (does random (tiebreak black)))
 ...
```

GDL-II Semantics

Mapping Game G to State-Transition model

- Σ set of all states S of ground atoms f
- $S^{\texttt{true}} = \{\texttt{true}(f_1), \cdots, \texttt{true}(f_n)\}$
  - S: set of ground atoms $f_1 \cdots f_n$
  - $S^{\texttt{true}}$: extension of S with `true` predicate
- $M^{\texttt{does}} = \{\texttt{does}(1, a_1), \cdots, \texttt{does}(r, a_r)\}$
  - $M^{\texttt{does}}$: joint move derivable from $G \cup S^{\texttt{true}}$
- Model $\mathcal{M} = (\Sigma, N, w_0, t, l, u, \mathcal{I}, g)$
  - $N = \{r \mid G \text{ satisfies } \texttt{role}(r)\}$
  - $w_0 = \{f \mid G \text{ satisfies } \texttt{init}(f)\}$
  - $u(M, S) = \{f \mid G \cup S^{\texttt{true}} \cup M^{\texttt{does}} \text{ satisfies } \texttt{next}(f)\}$ for all M and S
  - $\mathcal{I} = \{(r, M, S, p) \mid G \cup S^{\texttt{true}} \cup M^{\texttt{does}} \text{ satisfies } \texttt{sees}(r, p)\}$ for all $r \neq \texttt{random}$, M and S

Krieg Tic-Tac-Toe State-Transition model (1/3)

Building up model $\mathcal{M} = (N, w_0, t, l, u, \mathcal{I}, g)$

$\{black, white\} \subseteq N$

```
(role black)
(role white)
(role random)
```

$\{cell(1,1,b), ..., cell(3,3,b)\} \in w_0$ as

```
(init (cell 1 1 b))
...
(init (cell 3 3 b))
```

Krieg Tic-Tac-Toe State-Transition model (2/3)

Building up model $\mathcal{M} = (N, w_0, t, l, u, \mathcal{I}, g)$

$u(\langle(1,1)^x, (3,3)^o\rangle, w_0) = \{cell(1,1,x), ..., cell(3,3,o)\}$ as

$G \cup w_0^{\texttt{true}} \cup \langle(1,1)^x, (3,3)^o\rangle^{\texttt{does}}$ satisfies `(next (cell 1 1 x))`

and

$G \cup w_0^{\texttt{true}} \cup \langle(1,1)^x, (3,3)^o\rangle^{\texttt{does}}$ satisfies `(next (cell 3 3 o))`

Remind that rules with `next` are applied

```
(<= next (cell ?r ?m ?n)
        (true (cell ?m ?n b))
        (does white (mark ?m ?n)))
        (does black (mark ?m ?n)))
        (does random (tiebreak ?r)))
```

Krieg Tic-Tac-Toe State-Transition model (3/3)

Building up model $\mathcal{M} = (N, w_0, t, l, u, \mathcal{I}, g)$

$(x, \langle(1,1)^x, (3,3)^o\rangle, w_1, \mathit{cell}(1,1,x)) \in \mathcal{I}$ as

$G \cup w_0^{\texttt{true}} \cup \langle(1,1)^x, (3,3)^o\rangle^{\texttt{does}}$ satisfies `(sees x (cell 1 1 x))`

Remind that

```
(<= sees ?r1 (cell ?m1 ?n1 ?r1)
     (true (cell ?m1 ?n1 b))
     (does ?r1 (mark ?m1 ?n1))
     (does ?r2 (mark ?m2 ?n2))
     (distinct ?m1 ?m2))
```

Notice that $(o, \langle(1,1)^x, (3,3)^o\rangle, w_1, \mathit{cell}(1,1,x)) \notin \mathcal{I}$ as

$G \cup w_0^{\texttt{true}} \cup \langle(1,1)^x, (3,3)^o\rangle^{\texttt{does}}$ does not satisfies `(sees o (cell 1 1 x))`

Reasoning for winning?

From Game Theory to Logic

- Key question in GT: can the player win?
- What is *best response*?
- What about rational behaviour and equilibrium?

van Benthem (2012)

Much of game theory is about the question whether strategic equilibria exist. But there are hardly any explicit languages for defining, comparing, or combining strategies.

GDL-based Strategic Reasoning

Focus on the representation of strategies

Extend GDL and build a player on that extension

- Connecting action and output: how to play?
  - Quantification over possible runs is compulsory
    Overall assessment of the game: what happened if, instead of playing a, b is played?
  - Priority over eligible actions
    if action a leads to win while action b leads to loose, action a should be chosen (if rational)
- Question: how to represent predefined library of strategies?

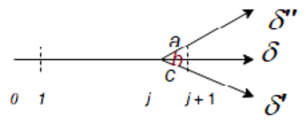
GDL-based Strategy Representation (1/5)

"Priority" operator: $\phi \nabla \psi$ [Jiang et al., 2014]

ϕ should hold; if not then ψ hold

$$M, \delta, j \models \phi \text{ or } (\texttt{Paths}(\phi, \delta[0, j]) = \emptyset \text{ and } M, \delta, j \models \psi)$$

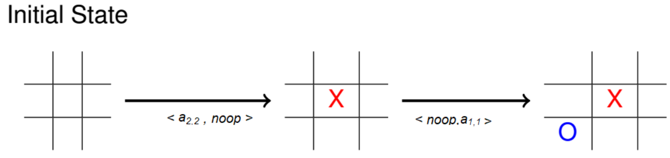
where $\texttt{Paths}(\phi, \delta[0, j])$ is the set of paths where ϕ holds at j and sharing initial segment $\delta[0, j]$:



$\texttt{Paths}(does(r, a), \delta[0, j]) = \{\delta''\}$ and $\texttt{Paths}(does(r, b), \delta[0, j]) = \{\delta\}$

GDL-based Strategy Representation (2/5)

Suppose M and δ:



- $M, \delta, 0 \models does(x, a_{2,2})$
- $M, \delta, 0 \not\models does(x, a_{1,3})$
- $M, \delta, 0 \models does(x, a_{2,2}) \triangledown does(x, a_{1,3})$
- $M, \delta, 1 \models does(o, a_{1,3})$
- $M, \delta, 1 \not\models does(o, a_{2,2})$
- $M, \delta, 1 \models does(o, a_{2,2}) \triangledown does(o, a_{1,3})$

GDL-based Strategy Representation (4/5)

Strategy rule

- syntax: $\phi := \varphi_1 \triangledown \varphi_2 \triangledown \cdots \triangledown \varphi_n$
- Non-ambiguous: at any state, ϕ must “elicit” only one action:
- Could be extended to perfect recall: consider history rather than state.
- Strategy for Player x (1st player)

  $combined^x := fill\_centre^x \triangledown check^x \triangledown block^x \triangledown fill\_corner^x \triangledown fill\_any^x$

  and

  $$\phi^x := (turn(x) \rightarrow combined^x) \wedge (\neg turn(x) \rightarrow noop^x)$$

- Strategy rule ϕ^x is a no loosing strategy for x

  No way to express the output in the GDL with priority

GDL-based Strategy Representation (5/5)

Example: strategy for Tic-Tac-Toe

- Fill the center:

  $$fill\_center^r = does(a^r_{2,2})$$

- Check if I can win:

  $$check^r = \bigvee_{i,j=1}^{3} (does(a^r_{i,j}) \wedge \bigcirc wins(r))$$

- Prevent immediate loss:

  $$block^r = \bigvee_{i,j=1}^{3} (\bigcirc(does(a^{-r}_{i,j}) \wedge \bigcirc wins(\text{-}r)) \wedge does(a^r_{i,j}))$$

- Fill an available corner:

  $$fill\_corner^r = \bigvee_{i,j \in \{1,3\}} does(a^r_{i,j})$$

- Fill anywhere available:

  $$fill\_any^r = \bigvee_{i,j=1}^{3} does(a^r_{i,j})$$

- Combined actions:

  $$combined^r = fill\_centre^r \triangledown check^r \triangledown block^r \triangledown fill\_corner^r \triangledown fill\_any^r$$

GDL-based Strategy Representation

A modal reading of the priority operator (1/2)

[Zhang and Thielscher, 2015]

- Basic GDL + look ahead operator: $\lfloor a \rfloor \varphi$

  If action a were chosen then φ would be true (but a is not executed)

- *does* operator restricted to joint action: does(a)
- New semantics relative to a state and a joint action: $w, a \models \varphi$
  - $w, a \models p$ iff $p \in \pi(w)$
  - $w, a \models does(b)$ iff $a = b$
  - $w, a \models \lfloor b \rfloor \varphi$ iff $w, b \models \varphi$

A modal reading of the priority operator (2/2)

- Prioritised disjunction operator

$$\varphi \nabla \psi =_{def} \varphi \vee (\psi \wedge \bigwedge_{c} \llcorner c \lrcorner \neg \varphi)$$

- In terms of semantics

  For any M, w and a: $w, a \models \varphi \nabla \psi$ iff either $w, a \models \varphi$ or $w, a \models \psi$ but $w, c \models \neg \varphi$ for all c

ATL-Reasoning about strategies

ATL for reasoning about GDL game description

- Use GDL game description as underlying semantic for ATL reasoning
- ATL: reasoning about cooperation

$$\langle\langle C \rangle\rangle \varphi \text{ Coalition } C \text{ can achieve } \varphi$$

- GDL + ATL:
  - check properties of game (playability)
  - check strategic properties

ATL Reasoning about strategies

Alternating-time Temporal Logic - Syntax

[Alur et al., 2002, Ruan et al., 2009]

- Coalition operator $\langle\langle C \rangle\rangle$
- Temporal operator $\bigcirc$ (next), $\Box$ (always), $\Diamond$ (sometimes), $\mathcal{U}$ (until)

$$\varphi \ ::= \ p \mid \varphi \vee \varphi \mid \langle\langle C \rangle\rangle \bigcirc \varphi \mid \langle\langle C \rangle\rangle \Box \varphi \mid \langle\langle C \rangle\rangle \Diamond \varphi \mid \langle\langle C \rangle\rangle \varphi \mathcal{U} \varphi$$

- coalition and temporal operators always together

$$\langle\langle x \rangle\rangle \Diamond wins(x) \vee \langle\langle x \rangle\rangle \Diamond \neg wins(-x)$$

ATL Reasoning about strategies

Alternating-time Temporal Logic - Semantics

- based on Concurrent Game Structure (or Transition systems)

$$\mathcal{A} = (\mathcal{Q}, q_0, N, \Pi, \pi, \mathit{legal}, \mathit{update})$$

  where
  - $\mathcal{Q}$: set of states
  - q_0: initial state
  - N: set of agents
  - Π: propositions
  - π: valuation function
  - *legal*: possible move function for each agent
  - *update*: deterministic joint move transition function
- Truth condition relative to a state q

$$\mathcal{A}, q \models_{ATL} \varphi$$

Alternating-time Temporal Logic - Semantics

- λ: sequence of states
- Additional component: strategy function $f_a(\lambda) \in \mathit{legal}(a, q)$ where q is the last state of λ

$$F_A = \{f_a | a \in A\}$$

- Output of a strategy: set of possible sequences $\lambda = q\, q'\, q'' ...$

$$\mathit{out}(q, F_A) = \{\lambda | \lambda[0] = q \text{ and}$$
$$\exists m \text{ s.t. } \forall a \in A, m_a \in f_a(\lambda[0..i]) \text{ and } (\lambda[i{+}1] = \mathit{update}(\lambda[i], m)\}$$

Alternating-time Temporal Logic - Semantics

- $\mathcal{A} = (\mathcal{Q}, q_0, N, \Pi, \pi, legal, update)$
- Truth conditions
  - $\mathcal{A}, q \models_{ATL} p$ iff $p \in \pi(q)$
  - $\mathcal{A}, q \models_{ATL} \langle\langle C \rangle\rangle \bigcirc \varphi$ iff there exists F_C such that:

    $$\mathcal{A}, \lambda[1] \models_{ATL} \varphi \text{ for all } \lambda \in out(q, F_C)$$

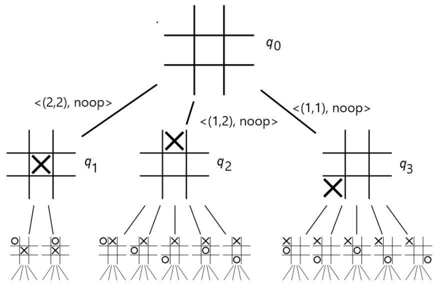
  - $\mathcal{A}, q \models_{ATL} \langle\langle C \rangle\rangle \Box \varphi$ iff there exists F_C such that:

    $$\mathcal{A}, \lambda[i] \models_{ATL} \varphi \text{ for all } \lambda \in out(q, F_C) \text{ and } i \geqslant 0$$

ATL Reasoning about strategies

$\mathcal{A} = (\mathcal{Q}, q_0, N, \Pi, \pi, legal, update)$

- Assume
  $f_x([q_0 q_1]) = noop$
- $\mathcal{A}, q_1 \models_{ATL}$
  $\langle\langle x \rangle\rangle \Box (p^x_{1,1} \vee p^x_{3,3})$
- Assume
  $f_o([q_0 q_2]) = \{(2,2)\}$
- $\mathcal{A}, q_2 \models_{ATL}$
  $\langle\langle o \rangle\rangle \bigcirc p^o_{2,2}$



ATL Reasoning about strategies

ATL for checking GDL specification

- Translation/embedding of GDL theory to ATL
- Model checking is EXPTIME
- Checking soundness

$$\langle\langle\rangle\rangle\Box((\mathit{terminal} \wedge \varphi) \rightarrow \langle\langle\rangle\rangle\Box(\mathit{terminal} \wedge \varphi))$$

- Winnable

$$\bigvee_i \langle\langle i\rangle\rangle\Diamond \mathit{wins}(i)$$

- Sequential

$$\langle\langle\rangle\rangle\Box(\langle\langle N\rangle\rangle \bigcirc \varphi \rightarrow \bigvee_i \langle\langle i\rangle\rangle \bigcirc \varphi)$$

ATL Reasoning about strategies

ATL for checking GDL specification

- Tic-Tac-Toe properties (CGS encoding)
- no-losing strategies for x

$$\langle\langle x \rangle\rangle \Box (terminal \rightarrow \neg wins(o))$$

- No explicit representation of actions (hidden in the semantics)

ATL Reasoning about strategies - On going work

Mixing priority and ATL operators (ongoing work)

- agent r may win?

$$posCheck^r = \bigvee_{i,j \in 1..3} does(r, a_{i,j}) \rightarrow \langle\langle r \rangle\rangle \Diamond check^r$$

- agent r can prevent $-r$ to win

$$posBlock^r = \bigvee_{i,j \in 1..3} does(r, a_{i,j}) \rightarrow \langle\langle r \rangle\rangle \Diamond block^r$$

(Towards) General strategic player

$$check^r \triangledown block^r \triangledown posCheck^r_a \triangledown posBlock^r_a$$

- Model checking is **EXPTIME**

GDL-based Strategy Representation - Going further

Pending questions:

- How to design strategies?
  Connection with Machine Learning and Planning
- Generalize strategies?
  Are they any common points (General Strategic Reasoning)
- How to implement?
  Complexity of strategic reasoning and complexity of the game

Still a lot to do! - Example: Equivalent games

Equivalent games (1/3)

Number Scrabble:

1. $initial \leftrightarrow turn(\mathrm{b}) \wedge \neg turn(\mathrm{w}) \wedge \bigwedge_{i=1}^{9} \neg(s(\mathrm{b}, i) \vee s(\mathrm{w}, i))$
2. $wins(r) \leftrightarrow \big(\bigvee_{i=2}^{3}(s(r,i) \wedge s(r,4) \wedge s(r, 11-i)) \vee$
   $\bigvee_{i=1}^{2}(s(r,i) \wedge s(r,6) \wedge s(r,9-i)) \vee$
   $\bigvee_{i=1}^{4}(s(r,5-i) \wedge s(r,5) \wedge s(r,5+i))\big)$
3. $terminal \leftrightarrow wins(\mathrm{b}) \vee wins(\mathrm{w}) \vee \bigwedge_{i=1}^{9}(s(\mathrm{b}, i) \vee s(\mathrm{w}, i))$
4. $legal(r, pick(n)) \leftrightarrow \neg(s(\mathrm{b}, n) \vee s(\mathrm{w}, n)) \wedge turn(r) \wedge \neg terminal$
5. $legal(r, noop) \leftrightarrow turn(-r) \vee terminal$
6. $\bigcirc s(r,n) \leftrightarrow s(r,n) \vee (\neg(s(\mathrm{b}, n) \vee s(\mathrm{w}, n)) \wedge does(r, pick(n)))$
7. $turn(r) \wedge \neg terminal \rightarrow \bigcirc \neg turn(r) \wedge \bigcirc turn(-r)$

Equivalent games (2/3)

Equivalence [Jiang et al., 2023]

Semantics 2 models (State-Transition) with a bisimulation between them

Syntax Set of rules are equivalent

Number Scrabble and Tic-Tac-Toe are equivalent

Equivalent Games (3/3)

Pending questions:

- Loose equivalence
  A game is “close ” to a second one? Restricted equivalence to a sub-part of the game?
- Connecting equivalence and strategic reasoning
  “ready-to-go” strategies
- How to implement
  Complexity for deciding whether two games are equivalent. Available heuristics?

Perspectives

A lot of questions!

On GDL:

- Connecting action and strategy
- Imperfect Information
- Games comparison

Still on GDL

- Connection to planning
- Construction of a General Player?
  Is it realistic to reason with GDL formulas?

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Want to work on it

PhD proposal: Compact Representation of Strategies

Starting date: Fall 2024

Keywords: strategic reasoning, GDL, ATL, Model-checking based player, Machine Learning.

More details: `laurent.perrussel@irit.fr`

Appendix

Proof theory of GDL

Mainly consists of axiom schemas for $\bigcirc$, modus ponens inference rule and general game properties:

Axioms

1. All tautologies of classical propositional logic.
2. $\bigcirc(\varphi \rightarrow \psi) \rightarrow (\bigcirc\varphi \rightarrow \bigcirc\psi)$
3. $\neg \bigcirc \varphi \rightarrow \bigcirc\neg\varphi$

Axioms for general game properties

4. $\neg \bigcirc \textit{initial}$
5. $\textit{terminal} \rightarrow \bigwedge_{a^r \in A^r \setminus \{noop^r\}} \neg legal(a^r) \wedge legal(noop^r)$
6. $\bigvee_{a^r \in A^r} does(r, a)$
7. $\neg(does(r, a) \wedge does(r, b))$ for $a^r \neq b^r$.
8. $does(a^r) \rightarrow legal(a^r)$
9. $\varphi \wedge \textit{terminal} \rightarrow \bigcirc\varphi$

GDL and Propositional Dynamic Logic (PDL)

- PDL formulas: $[\alpha]\varphi$ s.t. $[\alpha]\varphi =_{def} \neg\langle\alpha\rangle\neg\varphi$
- α limited to atomic program and sequence
- Interpretation over Kripke structure $M = (W, R_\alpha, v)$
- PDL semantics
    - $M, w \models p \iff p \in v(w)$
    - $M, w \models [\alpha]\varphi$ iff for all $w' \in R_\alpha$, $M, w' \models \varphi$

GDL and other logical systems

GDL and Propositional Dynamic Logic (PDL)

- Mapping between GDL and PDL
- First step: map the signature and formulas
- Second step: map the model (interpretations and paths)
- Third step: mapping result

$$M_{GDL}, \delta_{GDL}, j \models_{GDL} \varphi \iff M_{PDL}, w_j \models_{PDL} tr(\varphi)$$

GDL and Propositional Dynamic Logic (PDL)

- Mapping between GDL and PDL
- First step: map the signature and formulas
- Second step: map the model (interpretations and paths)
- Third step: mapping result

$$M_{GDL}, \delta_{GDL}, j \models_{GDL} \varphi \iff M_{PDL}, w_j \models_{PDL} tr(\varphi)$$

Epistemic extension: Axiomatics (1/3)

Mainly consists of axiom schemas and inference rules for $\bigcirc, \mathsf{K}_r, \mathsf{C}$ and general game properties [Jiang et al., 2017]

Axioms

1. All tautologies of classical propositional logic.

Axioms for general game properties

2. $\neg \bigcirc \mathit{initial}$
3. $\mathit{terminal} \rightarrow \bigwedge_{a^r \in A^r \setminus \{noop^r\}} \neg \mathit{legal}(a^r) \wedge \mathit{legal}(noop^r)$
4. $\bigvee_{a^r \in A^r} \mathit{does}(a^r)$
5. $\neg(\mathit{does}(a^r) \wedge \mathit{does}(b^r))$ for $a^r \neq b^r$.
6. $\mathit{does}(a^r) \rightarrow \mathit{legal}(a^r)$
7. $\varphi \wedge \mathit{terminal} \rightarrow \bigcirc\varphi$

Epistemic extension: Axiomatics (2/3)

Axioms for $\bigcirc, \mathsf{K}_r, \mathsf{C}$

8. $\bigcirc(\varphi \rightarrow \psi) \rightarrow (\bigcirc\varphi \rightarrow \bigcirc\psi)$
9. $\neg \bigcirc \varphi \leftrightarrow \bigcirc\neg\varphi$
10. $\mathsf{K}_r(\varphi \rightarrow \psi) \rightarrow (\mathsf{K}_r\varphi \rightarrow \mathsf{K}_r\psi)$
11. $\mathsf{K}_r\varphi \rightarrow \varphi$
12. $\mathsf{K}_r\varphi \rightarrow \mathsf{K}_r\mathsf{K}_r\varphi$
13. $\neg\mathsf{K}_r\varphi \rightarrow \mathsf{K}_r\neg\mathsf{K}_r\varphi$
14. $\mathsf{E}\varphi \leftrightarrow \bigwedge_{r=1}^{m} \mathsf{K}_r\varphi$
15. $\mathsf{C}\varphi \rightarrow \mathsf{E}(\varphi \wedge \mathsf{C}\varphi)$

Inference Rules

(R1) From φ, $\varphi \rightarrow \psi$ infer ψ.

(R2) From φ infer $\bigcirc\varphi$.

(R3) From φ infer $\mathsf{K}_r\varphi$.

(R4) From $\varphi \rightarrow \mathsf{E}(\varphi \wedge \psi)$ infer $\varphi \rightarrow \mathsf{C}\psi$.

Epistemic extension: Axiomatics (3/3)

Derivation about Krieg-Tic-Tac-Toe (full description: Σ_{KT}).

Proposition

For any $r \in N_{KT}$ *and* $a^r_{i,j} \in A^r_{KT}$,

1. $\vdash_{\Sigma_{KT}} initial \rightarrow \mathsf{C} initial$
2. $\vdash_{\Sigma_{KT}} legal(a^r_{i,j}) \rightarrow \mathsf{K}_r(legal(a^r_{i,j}))$
3. $\vdash_{\Sigma_{KT}} does(a^r_{i,j}) \rightarrow \bigcirc \mathsf{K}_r(p^r_{i,j} \vee tried(a^r_{i,j}))$
4. $\vdash_{\Sigma_{KT}} \mathsf{K}_r tried(a^r_{i,j}) \rightarrow \mathsf{K}_r p^{-r}_{i,j}$

Completeness... in one slide

Overall picture