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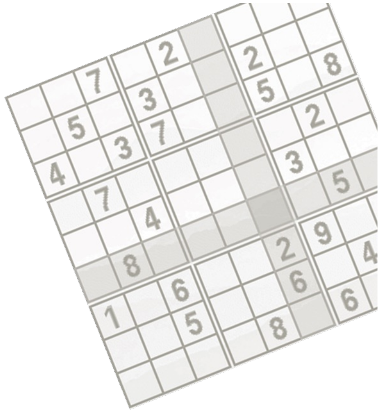

Introduction to Constraint Satisfaction

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Problem formulation, historical overview, applications

Sudoku?

- **Logic-based puzzle,** whose goal is to enter digits 1-9 in cells of 9×9 table in such a way, that no digit appears twice or more in every row, column, and 3×3 sub-grid.

9	6	3	1	7	4	2	5	8
1	7	8	3	2	5	6	4	9
2	5	4	6	8	9	7	3	1
8	2	1	4	3	7	5	9	6
4	9	6	8	5	2	3	1	7
7	3	5	9	6	1	8	2	4
5	8	9	7	1	3	4	6	2
3	1	7	2	4	6	9	8	5
6	4	2	5	9	8	1	7	3

A bit of history

1979: first published in New York

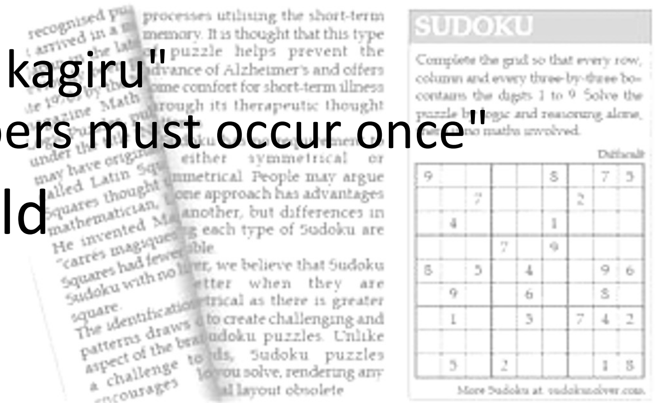
under the name „Number Place“

1986: became popular in Japan

Sudoku – from Japanes "**Su**dji wa **doku**shin ni kagiru"

"the numbers must be single" or "the numbers must occur once"

2005: became popular in the western world



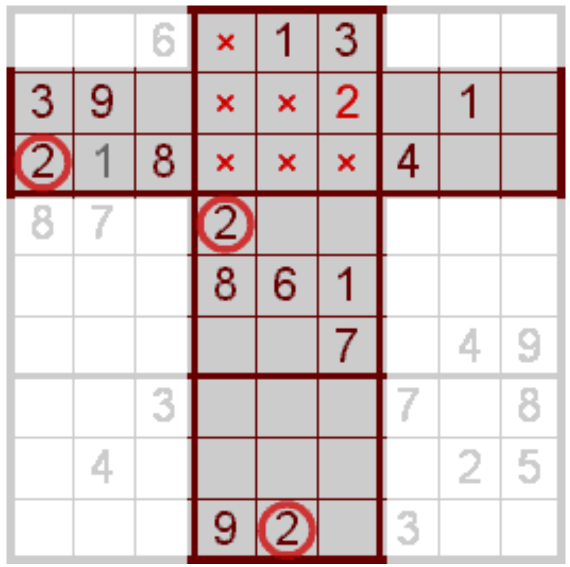
How to find out which digit to fill in?

- Use information that each digit appears exactly once in each row and column.

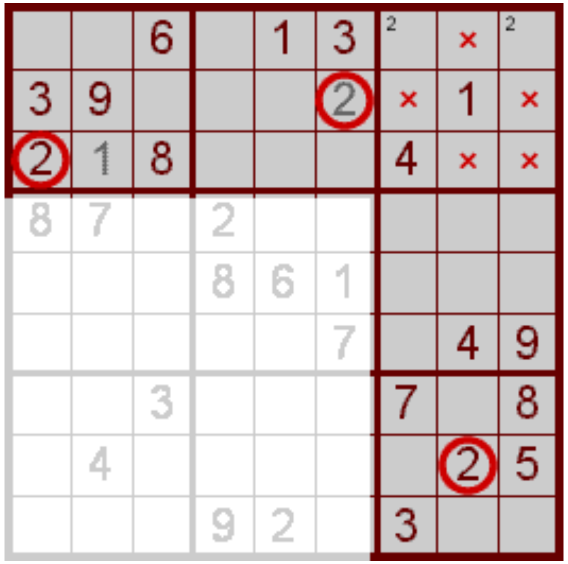
What if this is not enough?

- Look at columns

  or combine information from rows and columns

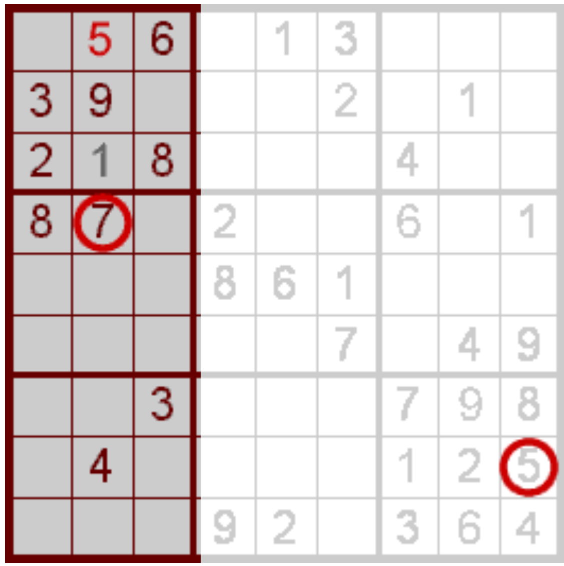


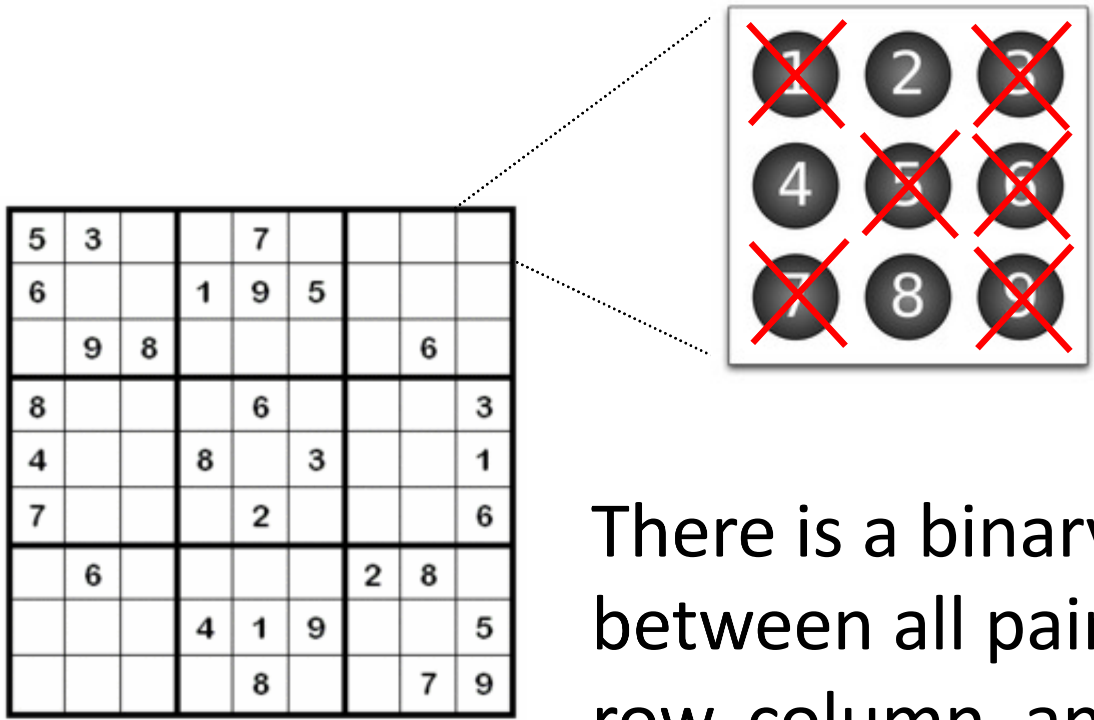
Sudoku – One More Step



- If neither rows and columns provide enough information, we can note allowed digits in each cell.

- The position of a digit cand be infereed from positions of other digits and resrictions of Sudoku that each digit appears one in a column (row, sub-grid)



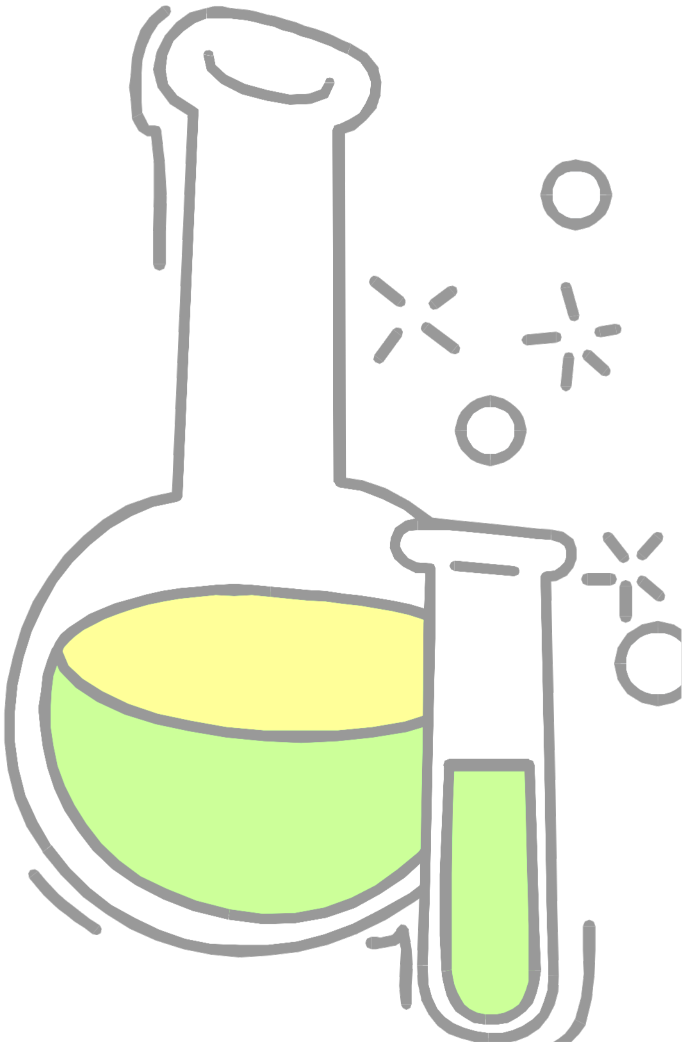


We can see every cell as a **variable** with possible values from **domain** {1,…,9}.

There is a binary inequality **constraint** between all pairs of variables in every row, column, and sub-grid.

Such formulation of the problem is called a **constraint satisfaction problem.**

1. **Introduction**: problem formulation, historical overview, applications.
2. **Search approaches**: local search (hill climbing, min-conflicts), depth-first search (backtracking, backjumping, backmarking).
3. **Local consistency techniques**: arc consistency and algorithms to achieve it (AC-3, AC-4).
4. **Higher-level consistency techniques**: path-consistency, k-consistency, global constraints.
5. **Integration** of consistency with search, value/variable ordering heuristics. **Optimization** problems. Problem **modelling**.



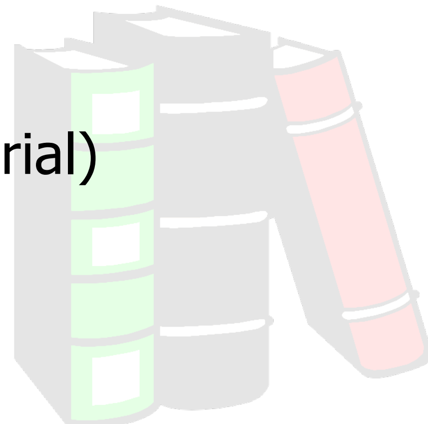
Resources

Books

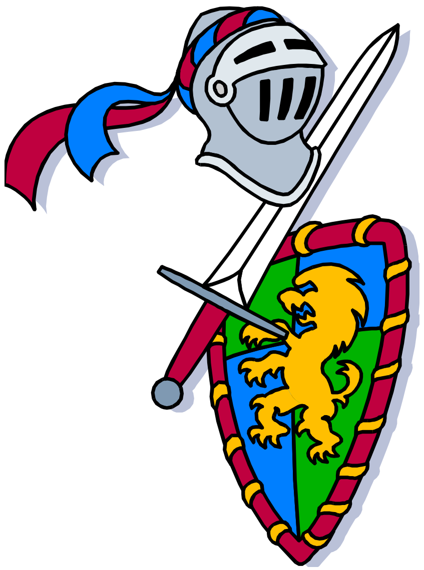
- P. Van Hentenryck: **Constraint Satisfaction in Logic Programming**, MIT Press, 1989
- E. Tsang: **Foundations of Constraint Satisfaction**, Academic Press, 1993
- K. Marriott, P.J. Stuckey: **Programming with Constraints: An Introduction**, MIT Press, 1998
- R. Dechter: **Constraint Processing**, Morgan Kaufmann, 2003
- **Handbook of Constraint Programming**, Elsevier, 2006

On-line resources

- **Charles University Course Web** (transparencies)
  http://ktiml.mff.cuni.cz/~bartak/podminky/
- **On-line Guide to Constraint Programming** (tutorial)
  http://ktiml.mff.cuni.cz/~bartak/constraints/

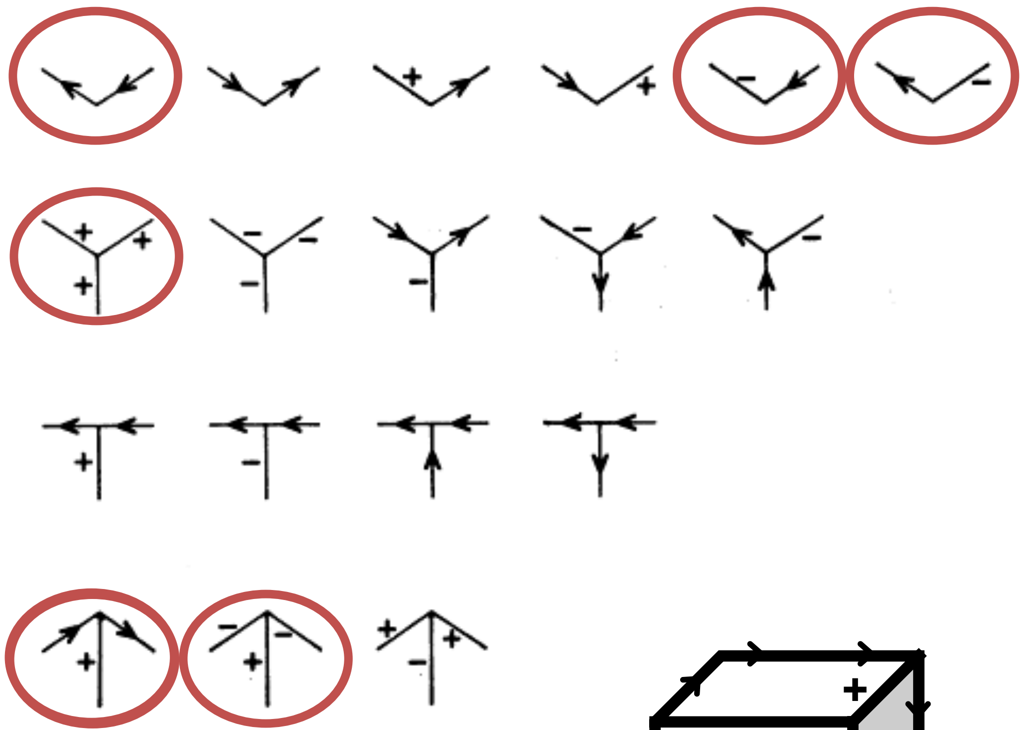


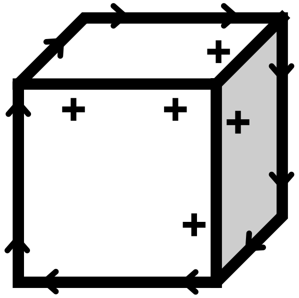
- **Artificial Intelligence**
  - Scene labelling (Waltz 1975)
  - How to help the search algorithm?
- **Interactive Graphics**
  - Sketchpad (Sutherland 1963)
  - ThingLab (Borning 1981)
- **Logic Programming**
  - unification $\rightarrow$ constraint solving (Gallaire 1985, Jaffar, Lassez 1987)
- **Operations Research and Discrete Mathematics**
  - NP-hard combinatorial problems

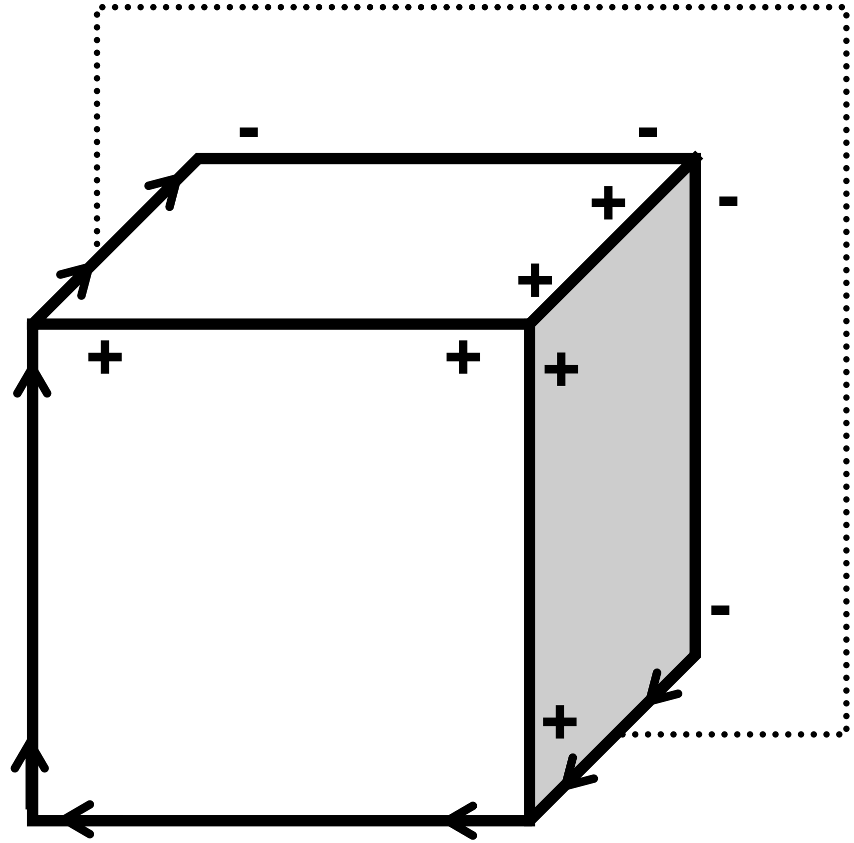


inferring 3D meaning of lines in a 2D drawing

- convex (+), concave (-) and border (←) edges
- we are looking for a physically feasible interpretation

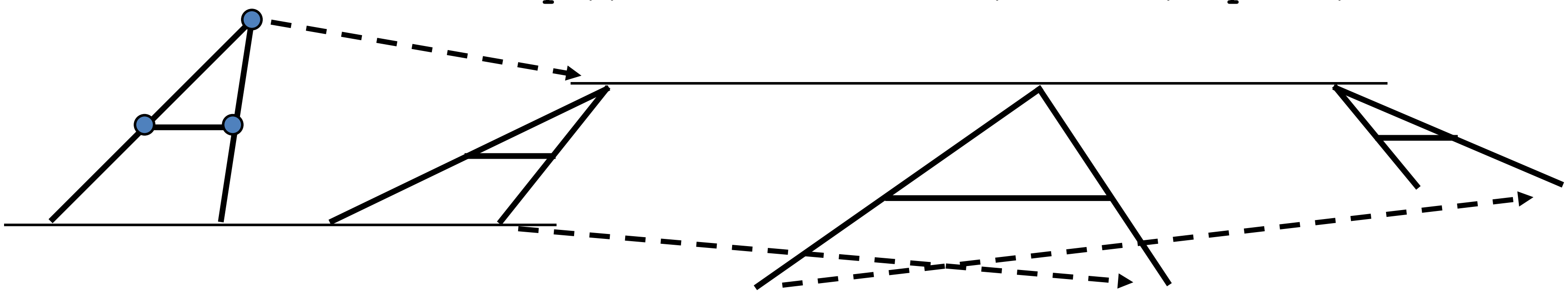


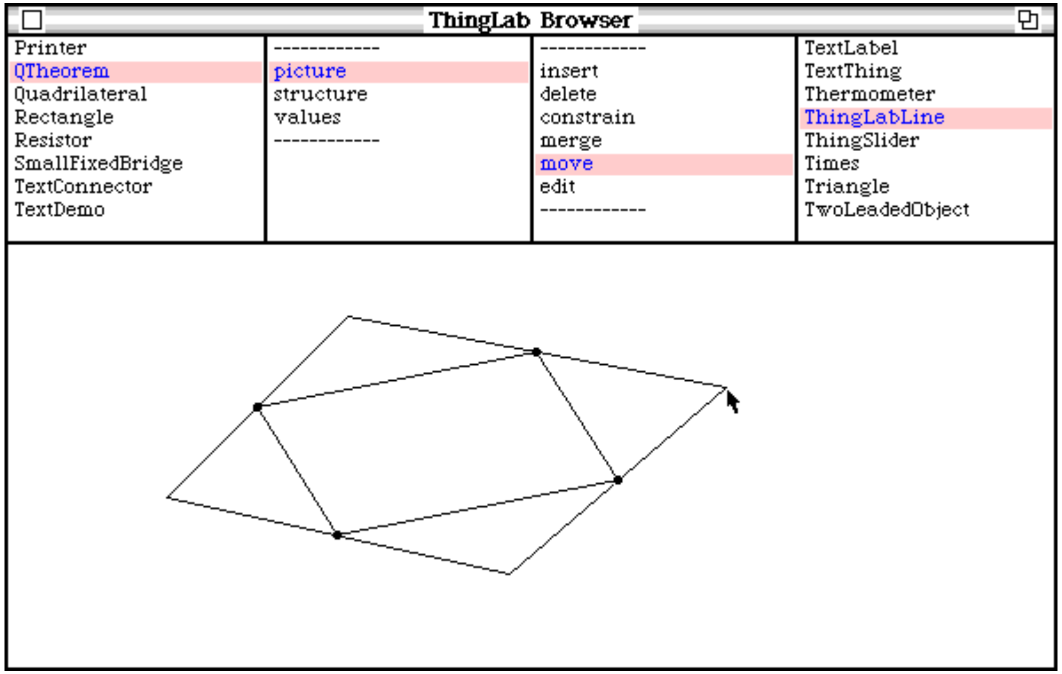


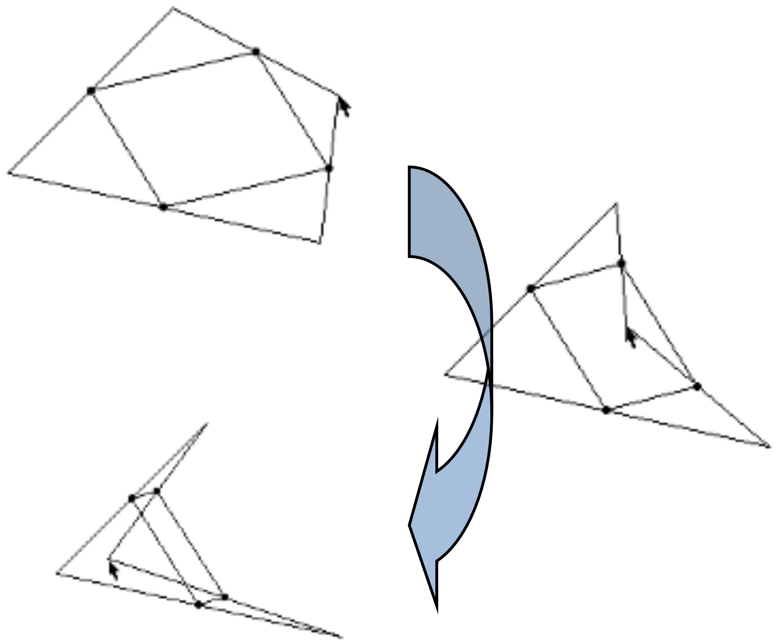


manipulating graphical objects described via constraints

http://ktiml.mff.cuni.cz/~bartak/diploma/downloads.html



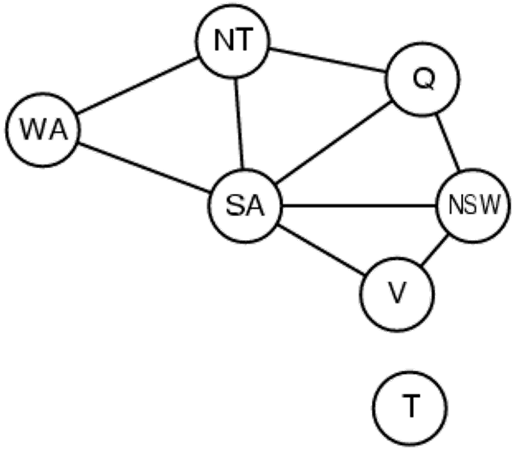




http://www.cs.washington.edu/research/constraints/

Map/Graph Colouring

Assign colours (red, blue, green) to states, such that neighbours have different colours.



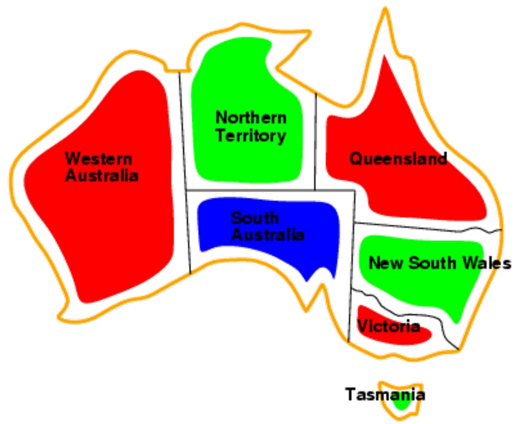
CSP Model

- variables: {WA, NT, Q, NSW, V, SA, T}
- domains: {r, b, g}
- constraints: WA ≠ NT, WA ≠ SA etc.

Can be described as a **constraint network** (nodes=variables, edges=constraints)

Solution

WA = r, NT = g, Q = r, NSW = g,
V = r, SA = b, T = g



Assign digits 0,…,9 to letters S,E,N,D,M,O,R,Y in such a way that:

- SEND + MORE = MONEY
- different letters are assigned to different digits
- S and M are different from 0

Model 1:

```
E,N,D,O,R,Y in 0..9, S,M in 1..9
            1000*S + 100*E + 10*N + D
+           1000*M + 100*O + 10*R + E
= 10000*M + 1000*O + 100*N + 10*E + Y
```

Model 2:

using „carry" 0-1 variables

```
E,N,D,O,R,Y in 0..9, S,M in 1..9, P1,P2,P3 in 0..1
   D+E = 10*P1+Y
P1+N+R = 10*P2+E
P2+E+O = 10*P3+N
P3+S+M = 10*M +O
```

```
all_different(S,E,N,D,M,O,R,Y)
```

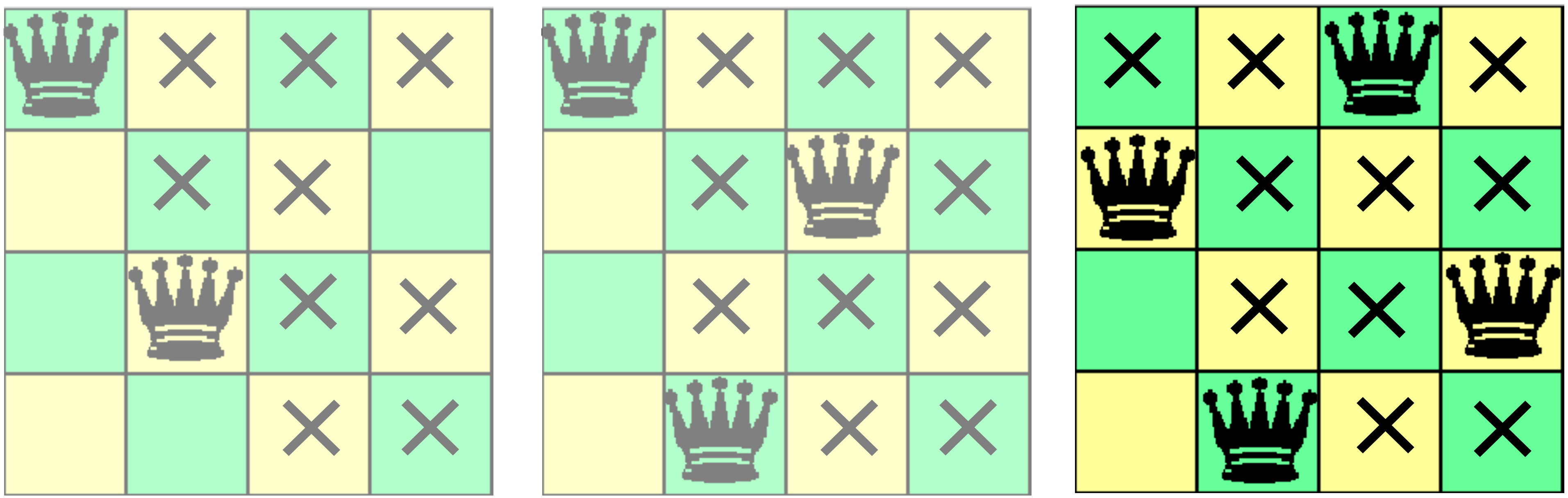
allocate N queens to a chess board of size N×N in a such way that no two queens attack each other

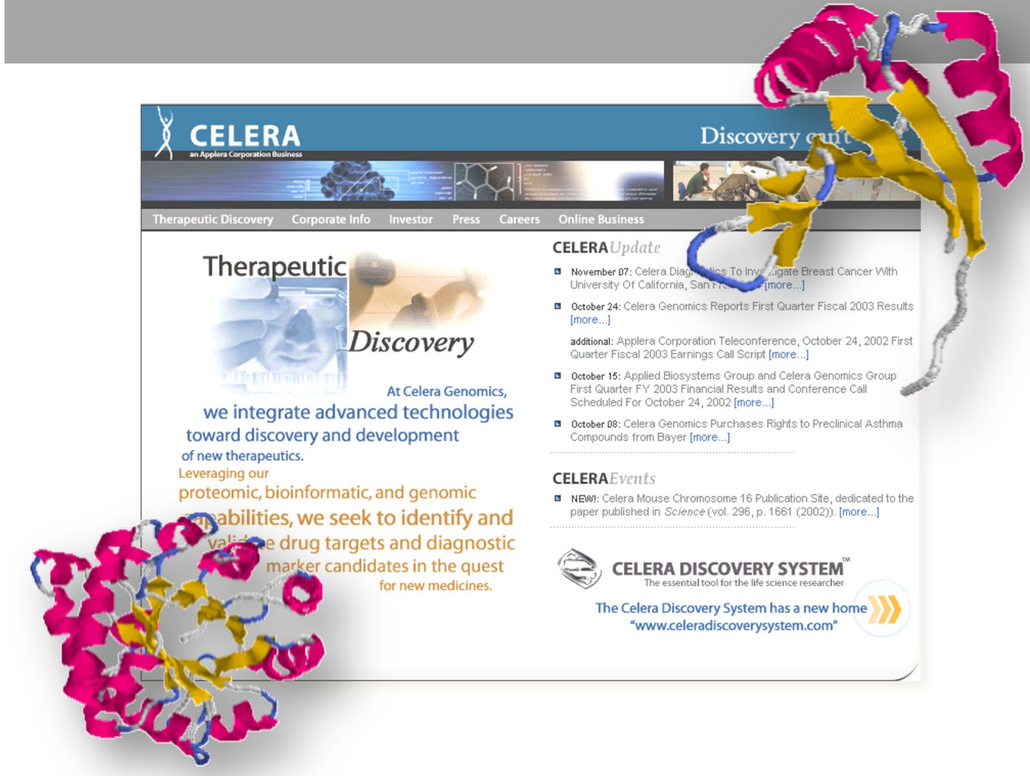
the core decision: each queen is located in its own column

variables: N variables r(i) with the domain {1,…,N}

constraints: no two queens attack each other

$$\forall i \neq j \quad r(i) \neq r(j) \wedge |i-j| \neq |r(i)-r(j)|$$



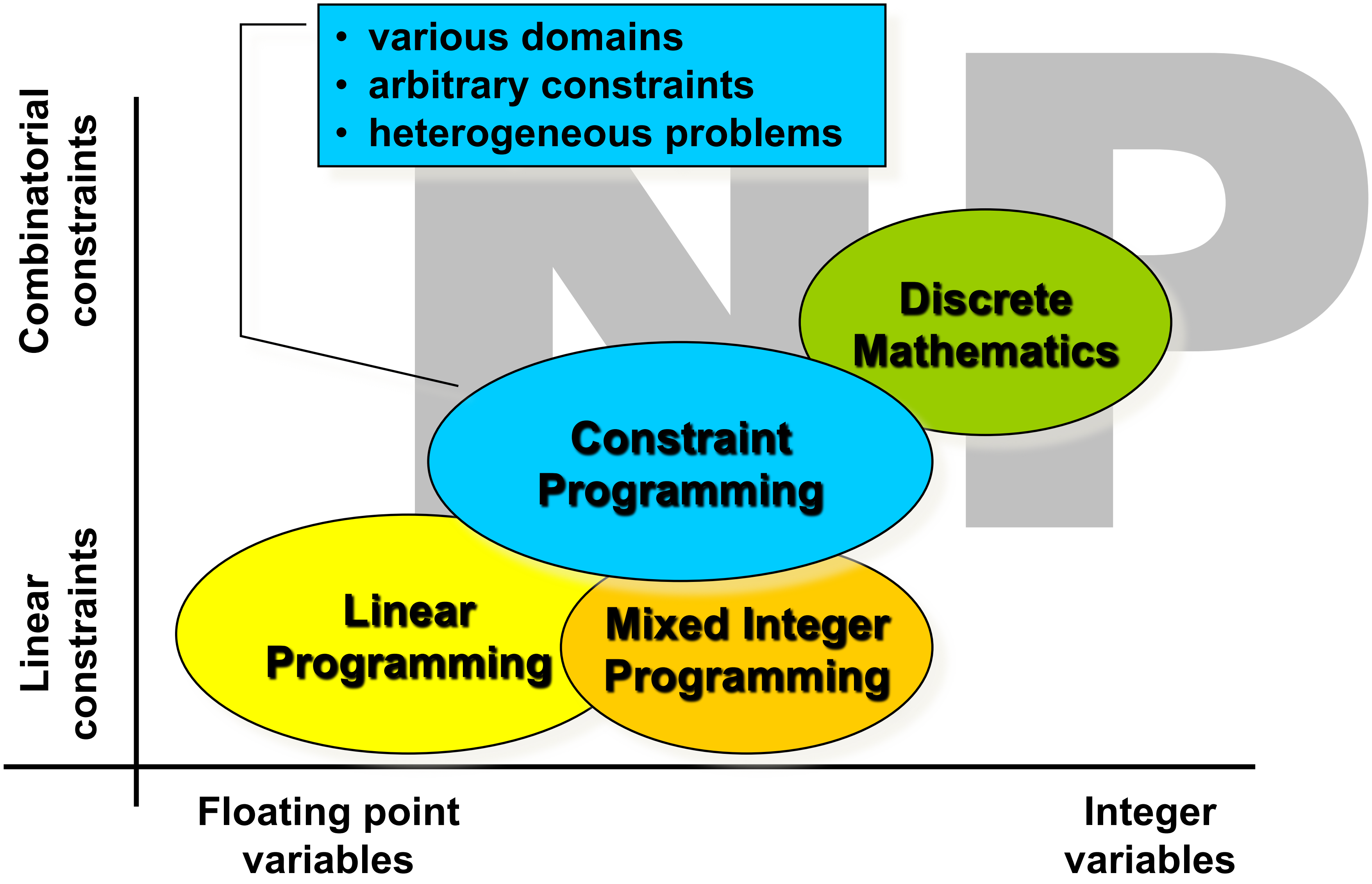


Bioinformatics

- DNA sequencing (Celera Genomics)
- deciding the 3D structure of proteins from the sequence of amino acids

Planning and Scheduling

- automated planning of spacecraft activities (Deep Space 1)
- manufacturing scheduling



• various domains
• arbitrary constraints
• heterogeneous problems
NP
Combinatorial constraints
Linear constraints
Discrete Mathematics
Constraint Programming
Linear Programming
Mixed Integer Programming
Floating point variables
Integer variables

Constraint Satisfaction Problem (CSP) consists of:

- a finite set of **variables**
  - describe attributes of the solution
    for example a location of a queen in the chess board
- **domains** – finite sets of possible values for variables
  - describe options that we need to decide
    for example, rows for queens
  - sometimes, there is a common super domain for all the variables and individual variables‘ domains are defined via unary constraints
- a finite set of **constraints**
  - constraint is a relation over a subset of variables
    for example locationA $\neq$ locationB
  - constraint can be defined in extension (a set of compatible value tuples) or using a formula (see above)

A feasible solution of a constraint satisfaction problem is a complete consistent assignment of values to variables.

- **complete** = each variable has assigned a value
- **consistent** = all constraints are satisfied

Sometimes we may look for all the feasible solutions or for the number of feasible solutions.

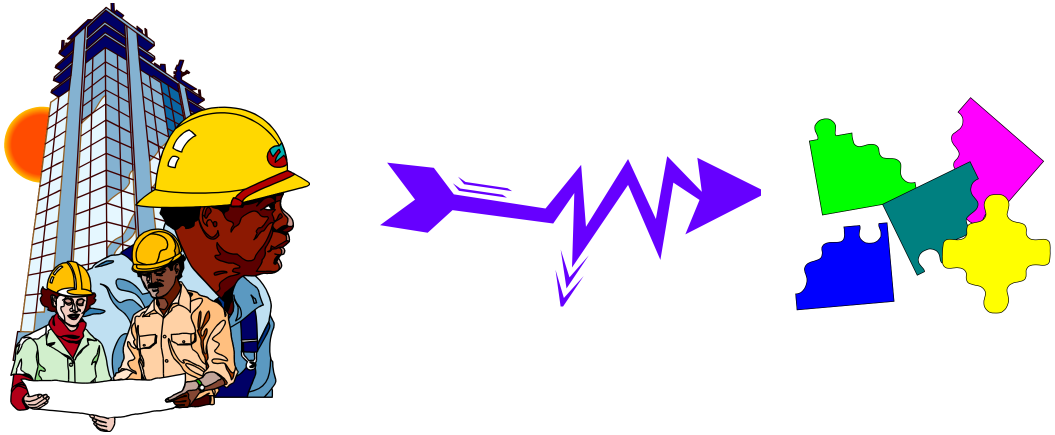
An optimal solution of a constraint satisfaction problem is a feasible solution that minimizes/maximizes a value of some objective function.

- **objective function** = a function mapping feasible solutions to real numbers

The Core Topics

Problem Modelling

How to describe a problem as a constraint satisfaction problem?



Solving Techniques

How to find values for the variables satisfying all the constraints?

Representation of constraints:

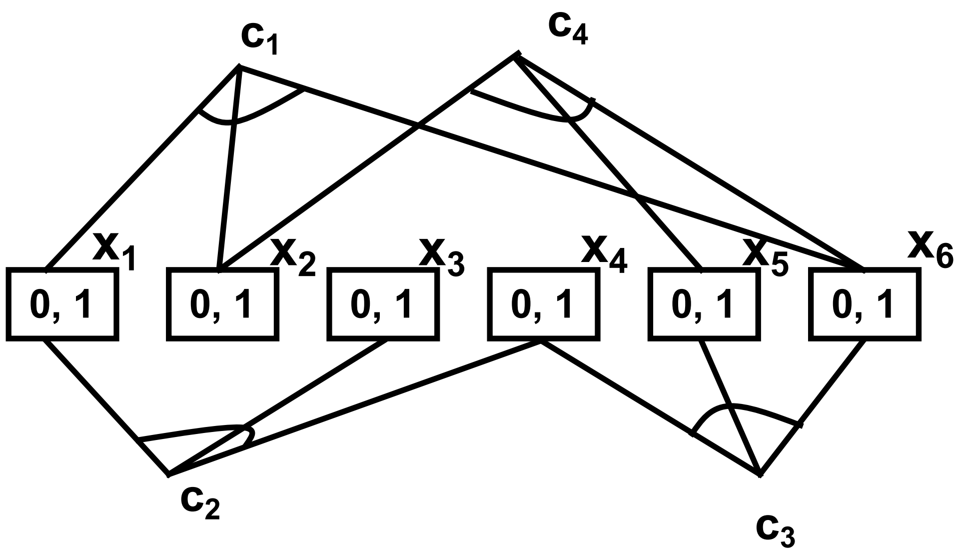
- intentional (algebraic/logic formulae)
- in extension (a set of compatible value tuples, 0-1 matrix)

Representation of a CSP as a (hyper)graph

- nodes = variables
- (hyper)edges = constraints

Example:

- variables $x_1,\ldots,x_6$ with domain {0,1}
- c_1: $x_1+x_2+x_6=1$
- c_2: $x_1-x_3+x_4=1$
- c_3: $x_4+x_5-x_6>0$
- c_4: $x_2+x_5-x_6=0$



The world is not binary ...

but it can be transformed to a binary one!

Binary CSP

CSP + all the constraints are binary

Note: unary constraints can be easily encoded in the domain of a variable

Equivalence of CSPs

Two constraint satisfaction problems are equivalent if they have the same sets of solutions.

Extended Equivalence of CSPs

Problem solutions can be syntactically transformed between the problems.

Can any CSP be transformed to an (extended) equivalent binary CSP?

Swapping variables and constraints.

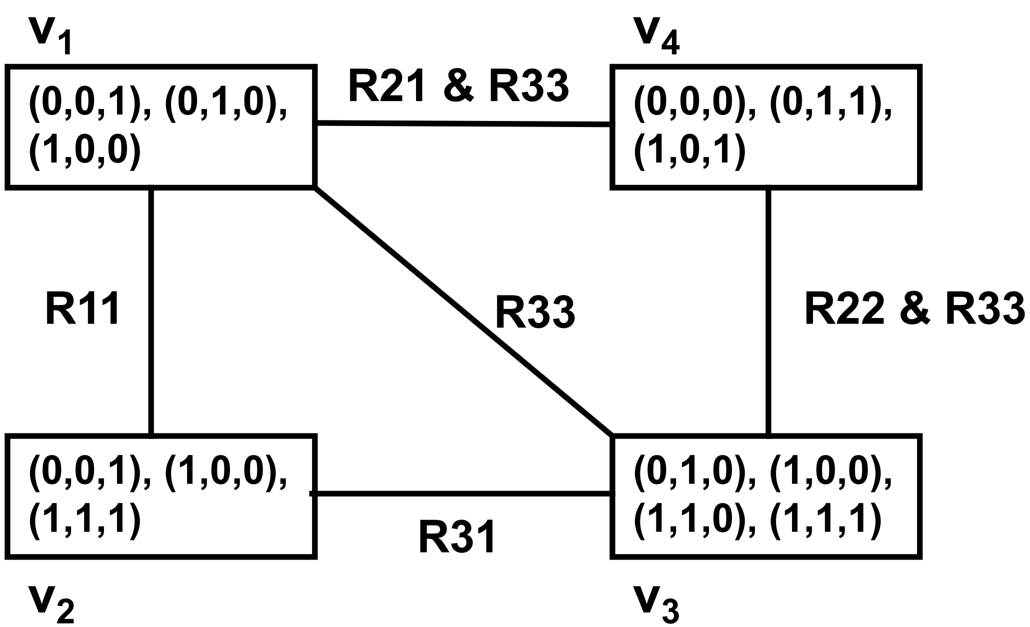
- k- ary constraint c is converted to a **dual variable** v_c with the domain consisting of compatible tuples

- for each pair of constraints c a c‘ sharing some variables there is a **binary constraint** between v_c a $v_{c'}$ restricting the dual variables to tuples in which the original shared variables take the same value

Example:

- variables $x_1,\ldots,x_6$ with domain {0,1}

- c_1: $x_1+x_2+x_6=1$
- c_2: $x_1-x_3+x_4=1$
- c_3: $x_4+x_5-x_6>0$
- c_4: $x_2+x_5-x_6=0$

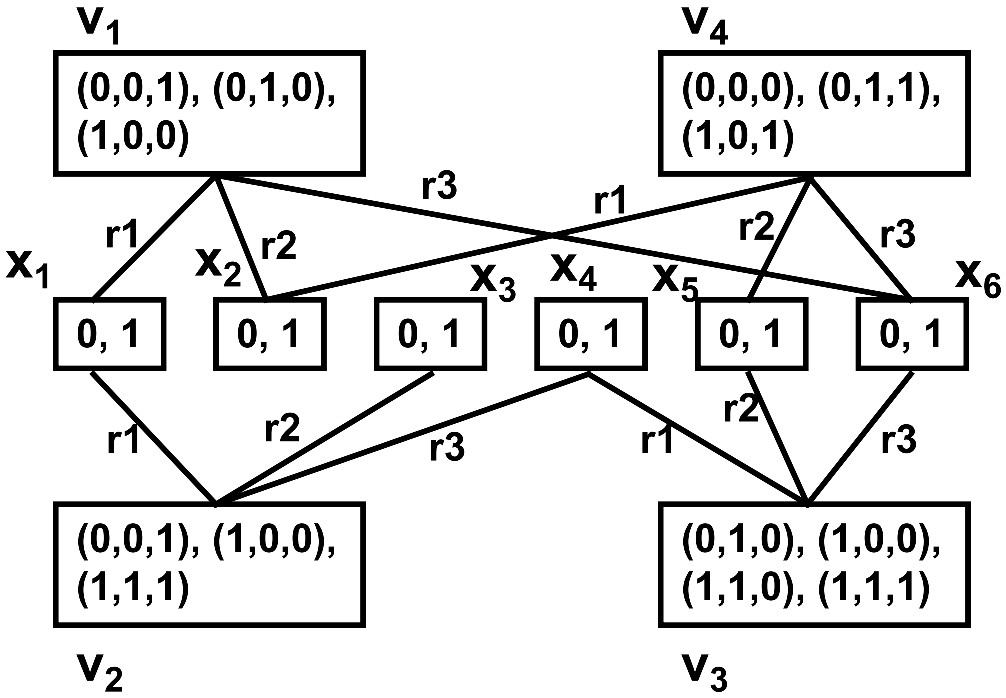


New dual variables for (non-binary) constraints.

- k- ary constraint c is translated to a **dual variable** v_c with the domain consisting of compatible tuples
- for each variable x in the constraint c there is a constraint between x a v_c restricting tuples of dual variable to be compatible with x

Example:

- variables $x_1,\ldots,x_6$ with domain {0,1}
- c_1: $x_1+x_2+x_6=1$
- c_2: $x_1-x_3+x_4=1$
- c_3: $x_4+x_5-x_6>0$
- c_4: $x_2+x_5-x_6=0$



Why do we do binarisation?

- a unified form of a CSP
- many solving approaches are formulated for binary CSPs
- tradition (historical reasons)

Which encoding is better?

- hard to say ;-)
- dual encoding:
  better propagation but constraints in extension
- hidden variable encoding:
  keeps original variables but weaker propagation

Binary vs non-binary constraints

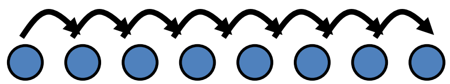
- more complex propagation algorithms for non-binary constraints
- exploiting semantics of constraints for more efficient and
  stronger domain filtering

Searching for a solution

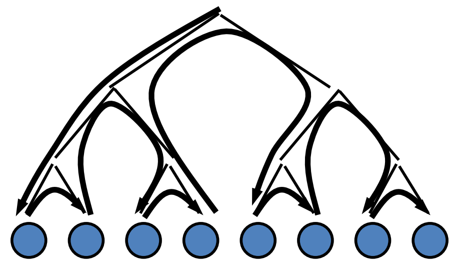
The goal: **find a complete and consistent instantiation of variables**

Two **core solving approaches**:

- **exploring complete but possibly inconsistent assignments** until a consistent assignment is found
  - generate and test, local search



- **extending a partial consistent assignment** until a complete assignment is reached
  - backtracking and its extensions

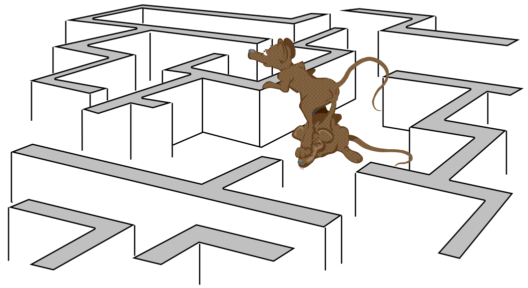


We can explore assignments in two ways:

- **systematically** (explore all possible assignments systematically)
  - a complete method, but could be too slow
- **non-systematically** (some assignments can be skipped)
  - an incomplete method, but can found solution much faster

Note:

We will use constraints in a *passive way*, just to verify whether the given assignment (even partial) satisfies the constraint.



Work plan:

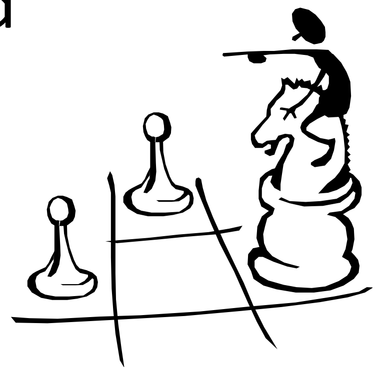
- start simple (with a trivial algorithm)
- find weaknesses of the algorithm
- repair the weaknesses to get better algorithms

In particular:

- start with **generate and test** method
- **improve the generator**
  - local search methods (HC, RW, TS, GSAT, GENET, SA)
- **merge the generator with the tester**
  - backtracking methods
  - improvements of chronological backtracking
    - backjumping, dynamic backtracking, backmarking

Probably the most general problem solving method

1) generate a candidate for solution
2) test if the candidate is really a solution

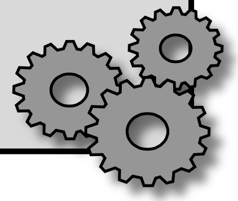


How to apply GT to CSP?

1) assign values to all variables
2) test whether all the constraints are satisfied

GT **explores complete but inconsistent assignments** until a (complete) consistent assignment is found.

procedure GT(X:variables, C:constraints)
    V ← construct a first complete assignment of X
    while V does not satisfy all the constraints C **do**
        V ← construct systematically a complete assignment next to V
    end while
    return V



The greatest weakness of GT is **exploring too many "visibly" wrong assignments**.

Example:

X::{1,2}, Y::{1,2}, Z::{1,2} $X = Y, X \neq Z, Y > Z$

X	1	1	1	1	2	2	2
Y	1	1	2	2	1	1	2
Z	1	2	1	2	1	2	1

How to improve GT?

- **smart generator**
  - the next assignment improves over the current assignment
  - the core idea of local search techniques
- **merged generate and test stages** (earlier detection of clash)
  - constraints are tested as soon as all involved variables are instantiated
  - backtracking

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