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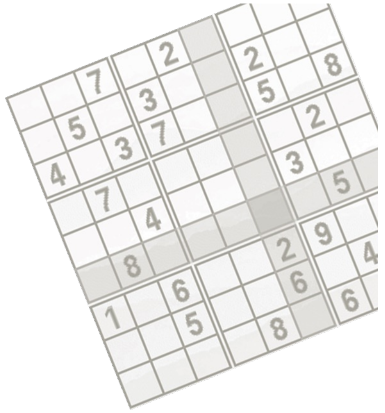

Introduction to Constraint Satisfaction

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Arc consistency

So far we used constraints in a passive way (as a test).

in the best case we analysed the reason of the conflict.

Can we use the constraints in a more active way?

Example:

A in 3..7, B in 1..5 the variables' domains

A<B the constraint

– many inconsistent values can be removed

– we get A in 3..4, B in 4..5

Note: it does not mean that all the remaining combinations of the values are consistent (for example A=4, B=4 is not consistent)

- How to remove the inconsistent values from the variables' domains in the constraint network?

Node consistency (NC)

Unary constraints are converted into variables' domains.

Definition:

- **The vertex** representing variable X is **node consistent** iff every value in the variable's domain D_x satisfies all the unary constraints imposed on the variable X.
- **CSP** is **node consistent** iff all the vertices are node consistent.

Algorithm NC

```
procedure NC(G)
    for each variable X in nodes(G)
        for each value V in the domain $D_X$
            if unary constraint on X is inconsistent with V then
                    delete V from $D_X$
        end for
    end for
end NC
```

Arc consistency (AC)

Since now we will assume binary CSPs only

i.e. a constraint corresponds to an arc (edge) in the constraint network.

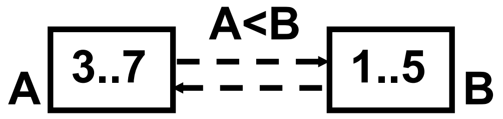
Definition:

- **The arc** *(V_i,V_j)* is **arc consistent** iff for each value *x* from the domain D_i there exists a value *y* in the domain D_j such that the assignment $V_i = x$ a $V_j = y$ satisfies all the binary constraints on V_i, V_j.

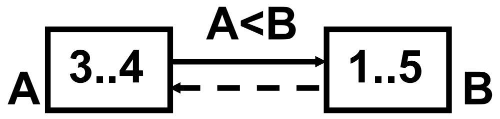
Note: The concept of arc consistency is directional, i.e., arc consistency of *(V_i,V_j)* does not guarantee consistency of *(V_j,V_i)*.

- **CSP** is **arc consistent** iff every arc *(V_i,V_j)* is arc consistent (in both directions).

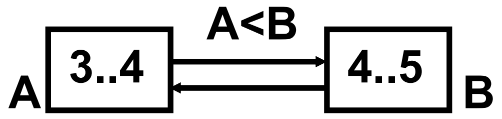
Example:



no arc is consistent



(A,B) is consistent



(A,B) and (B,A) are consistent

Algorithm for arc revisions

How to make (V_i,V_j) arc consistent?

- Delete all the values *x* from the domain D_i that are inconsistent with all the values in D_j (there is no value *y* in D_j such that the valuation $V_i = x$, $V_j = y$ satisfies all the binary constraints on V_i a V_j).

Algorithm of arc revision

```
procedure REVISE((i,j))
    DELETED ← false
    for each X in $D_i$ do
        if there is no such Y in $D_j$ such that (X,Y) is consistent, i.e.,
                (X,Y) satisfies all the constraints on $V_i$, $V_j$ then
            delete X from $D_i$
            DELETED ← true
        end if
    end for
    return DELETED
end REVISE
```

The procedure also reports the deletion of some value.

Algorithm AC-1

How to make a CSP arc consistent?

- Do revision of every arc.

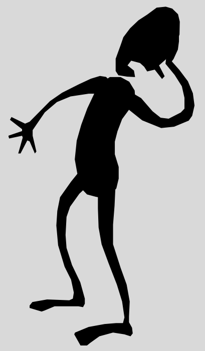
Beware, this is not enough! Pruning the domain may make some already revised arcs inconsistent again.

A<B, B<C: $(\underline{3..7},\mathit{1..5},1..5)$ $(\mathit{3..4},\underline{1..5},1..5)$ $(3..4,\underline{4..5},\mathit{1..5})$ $(3..4,\mathit{4},\underline{1..5})$ $(\underline{3..4},\mathit{4},5)$ $(3,4,5)$

- Thus the arc revisions will be repeated until any domain is changed.

Algorithm AC-1

```
procedure AC-1(G)
    repeat
        CHANGED ← false
        for each arc (i,j) in G do
            CHANGED ← REVISE((i,j)) or CHANGED
        end for
    until not(CHANGED)
end AC-1
```



What is wrong with AC-1?

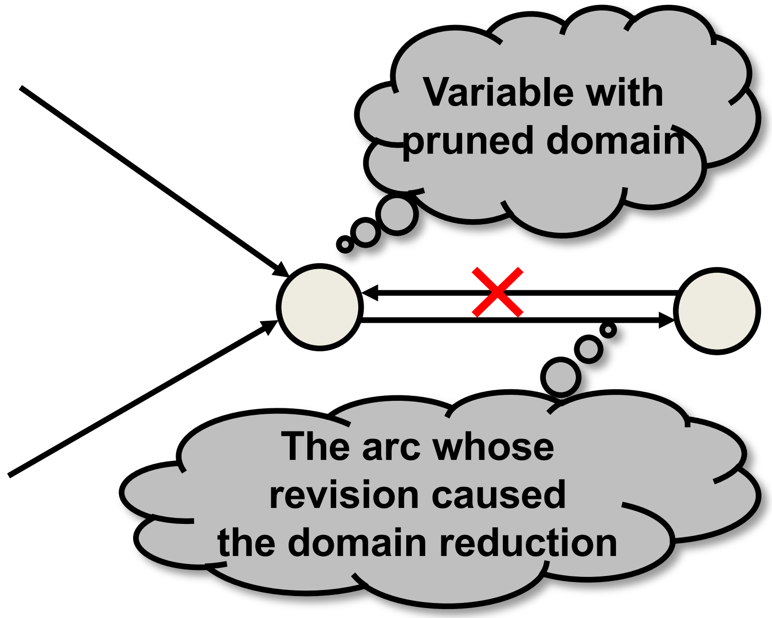
- If a single domain is pruned then revisions of all the arcs are repeated even if the pruned domain does not influence most of these arcs.

Which arcs should be reconsidered for revisions?

- The arcs whose consistency is affected by the domain pruning, i.e., the arcs pointing to the changed variable.

We can omit one more arc!

Omit the arc running out of the variable whose domain has been changed (this arc is not affected by the domain change).


Variable with pruned domain
The arc whose revision caused the domain reduction

Algorithm AC-2

A generalised version of the Waltz's labelling algorithm.

- In every step, the arcs going back from a given vertex are processed (i.e. a sub-graph of visited nodes is AC)

Algorithm AC-2

```
procedure AC-2(G)
    for i ← 1 to n do                                   % n is a number of variables
        Q ← {(i,j) | (i,j)∈arcs(G), j<i}                % arcs for the base revision
        Q' ← {(j,i) | (i,j)∈arcs(G), j<i}               % arcs for re-revision
        while Q non empty do
            while Q non empty do
                select and delete (k,m) from Q
                if REVISE((k,m)) then
                    Q' ← Q' ∪ {(p,k) | (p,k)∈arcs(G), p≤i, p≠m }
            end while
            Q ← Q'
            Q' ← empty
        end while
    end for
end AC-2
```

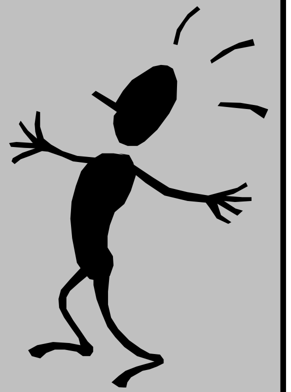
Algorithm AC-3

Re-revisions can be done more elegantly than in AC-2.

1. one queue of arcs for (re-)revisions is enough
2. only the arcs affected by domain reduction are added to the queue (like AC-2)

Algorithm AC-3

```
procedure AC-3(G)
    Q ← {(i,j) | (i,j)∈arcs(G), i≠j}      % queue of arcs for revision
    while Q non empty do
        select and delete (k,m) from Q
        if REVISE((k,m)) then
            Q ← Q ∪ {(i,k) | (i,k)∈arcs(G), i≠k, i≠m}
        end if
    end while
end AC-3
```

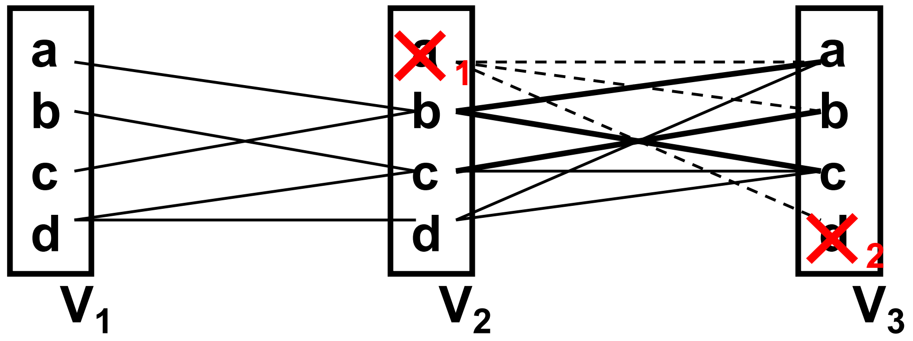


AC-3 is the most widely used consistency algorithm but it is still not optimal.

Observation (AC-3):

– Many pairs of values are tested for consistency in every arc revision.

– These tests are repeated every time the arc is revised.



1. When the arc V_2,V_1 is revised, the value *a* is removed from domain of V_2.

2. Now the domain of V_3, should be explored to find out if any value *a,b,c,d* loses the support in V_2.

Observation:

The values *a,b,c* need not be checked again because they still have supports in V_2 different from *a*.

The support set for $a \in D_i$ is the set $\{<j,b> \mid b \in D_j\ , (a,b) \in C_{i,j}\}$

Can we compute the support sets once and then use them during re-revisions?

Finding support sets

- A set of values supported by a given value (if the value disappears then these values lost one support), and a number of own supports are kept.

Computing and counting supporters

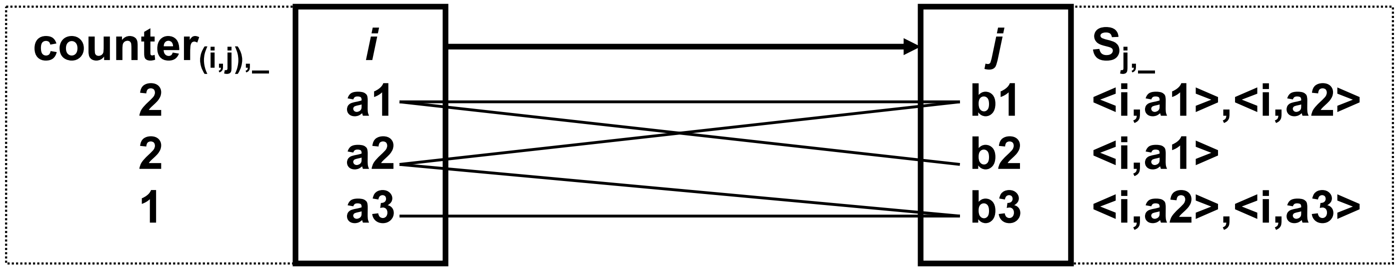
```
procedure INITIALIZE(G)
    $Q \leftarrow \{\}$ , $S \leftarrow \{\}$                % emptying the data structures
    for each arc $(V_i,V_j)$ in arcs(G) do
        for each a in $D_i$ do
            total $\leftarrow$ 0
            for each b in $D_j$ do
                if (a,b) is consistent according to the constraint $C_{i,j}$ then
                        total $\leftarrow$ total + 1
                        $S_{j,b} \leftarrow S_{j,b} \cup \{<i,a>\}$
                end if
            end for
            counter[(i,j),a] $\leftarrow$ total
            if counter[(i,j),a] = 0 then
                delete a from $D_i$
                $Q \leftarrow Q \cup \{<i,a>\}$
            end if
        end for
    end for
    return Q
end INITIALIZE
```

$S_{j,b}$ **- a set of pairs <i,a> such that <j,b> supports them**

counter[(i,j),a] **- number of supports for the value *a* from D_i in the variable V_j**

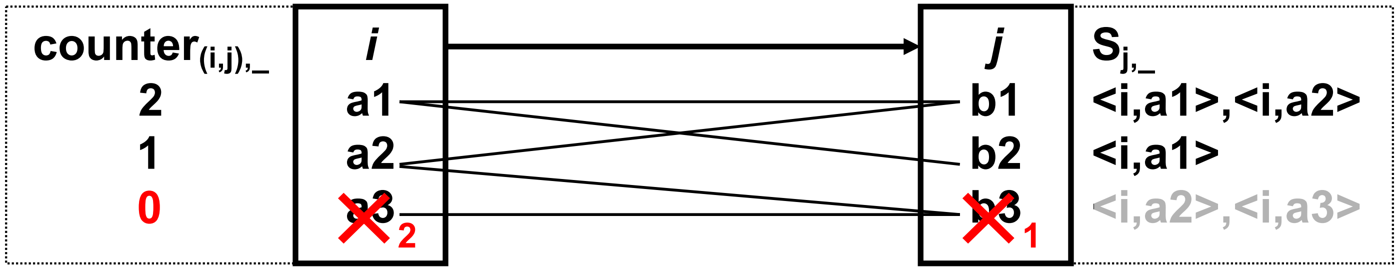
Situation:

we have just processed the arc (i,j) in INITIALIAZE



Using the support sets:

1. Let b3 is deleted from the domain of j (for some reason).
2. Look at $S_{j,b3}$ to find out the values that were supported by b3 (i.e. <i,a2>,<i,a3>).
3. Decrease the counter for these values (i.e. tell them that they lost one support).
4. If any counter becomes zero (a3) then delete the value and repeat the procedure with the respective value (i.e., go to 1).

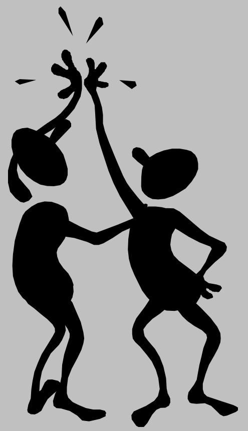


Algorithm AC-4

The algorithm AC-4 has the optimal worst-case time complexity!

Algorithm AC-4

```
procedure AC-4(G)
    Q ← INITIALIZE(G)
    while Q non empty do
        select and delete any pair <j,b> from Q
        for each <i,a> from $S_{j,b}$ do
            counter[(i,j),a] ← counter[(i,j),a] - 1
            if counter[(i,j),a] = 0 & "a" is still in $D_i$ then
                    delete "a" from $D_i$
                    Q ← Q ∪ {<i,a>}
            end if
        end for
    end while
end AC-4
```



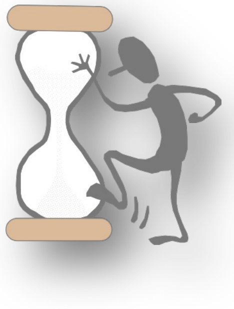
Unfortunately the average efficiency is not so good

… plus there is a big memory consumption!

Other arc consistency algorithms

- **AC-5 (Hentenryck, Deville, Teng 1992)**
  - a generic arc-consistency algorithm
  - can be reduced both to AC-3 and AC-4
  - exploits semantic of the constraint
  - functional, anti-functional, and monotonic constraints
- **AC-6 (Bessiere 1994)**
  - improves memory complexity and average time complexity of AC-4
  - keeps one support only, the next support is looked for when the current support is lost
- **AC-7 (Bessiere, Freuder, Regin 1999)**
  - based on computing supports (like AC-4 and AC-6)
  - exploits symmetry of the constraint

AC-3.1: optimal AC-3



Some observations:

- AC-3 is not (theoretically) optimal
- AC-4 is (theoretically) optimal but (practically) slow
- AC-6/7 are (practically) faster than AC-4, but quite complicated

What is inefficient in AC-3?

- Looking for supports in REVISE starts from scratch!

if „there is no such Y in D_j such that (X,Y) is consistent“ **then**

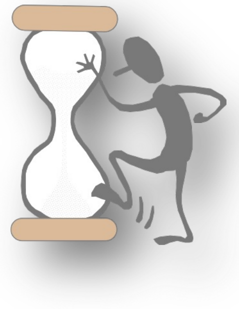
AC-3.1

- same run as AC-3
- but for each value, it remembers the last support in the constraint and the next time, it starts looking for a support at this value

```
procedure EXIST((i,x),j)
      y ← last((i,x),j)
      if y∈D_j then return true
      while y←next(y,D_j) & y≠nil do
            if (x,y)∈C(i,j) then
                  last((i,x),j) ← y
                  return true
      end while
      return false
end EXIST
```

Algorithm AC-2001: another optimal AC-3

Version of AC-3 with the queue of variables (AC-8)



```
procedure AC-2001(G)
    $Q \leftarrow \{i \mid i \in nodes(G)\}$   % a queue of nodes for revision
    while Q non empty do
        select and delete j from Q
        for each $i \in nodes(G)$ such that $(i,j) \in arcs(G)$ do
            if REVISE2001(i,j) then
                if $D_i = \emptyset$ then return fail
                $Q \leftarrow Q \cup \{i\}$
        end for
    end while
    return true
end AC-2001
```

```
procedure REVISE2001(i,j)
    DELETED ← false
    for each x in $D_i$ do
        if $last((i,x),j) \notin D_j$ then
            if $\exists y \in D_j\ y > last((i,x),j)$
            & $(x,y) \in C(i,j)$ then
                $last((i,x),j) \leftarrow y$
            else
                delete x from $D_i$
                DELETED ← true
            end if
    end for
    return DELETED
end REVISE2001
```

Note:

- The algorithm works with difference sets – for each variable we know a set of values deleted since the last revision.

Directional arc consistency

Observation 1:

AC has a directional character but a CSP is not directional.

Observation 2:

AC has to repeat arc revisions and the number of revisions depends on the number of arcs and on the domain size (the while loop).

Can we weaken AC somehow so each arc is revised exactly once?

Definition:

CSP is **directional arc consistent** for a given order of variables if and only if each arc (i,j), such that i<j, is consistent.

Again, each arc is checked once, but only in a one direction.

Algorithm DAC-1

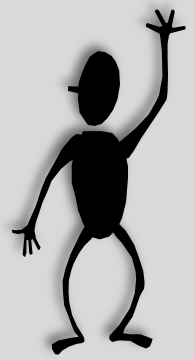
1. Arc consistency is required in one direction only
2. Variables are ordered
   - ↳ no directed cycle!

$$\begin{tikzcd}
1 \arrow[r, bend left, "6"] \arrow[rr, bend right, "5"'] & 2 \arrow[rr, bend left, "4"] & 3 \arrow[r, bend left, "3"] \arrow[rr, bend right, "2"'] & 4 \arrow[r, bend left, "1"] & 5
\end{tikzcd}$$

If arcs are revised in the right order then no revision needs to be repeated!

Algorithm DAC-1

```
procedure DAC-1(G)
    for j = |nodes(G)| to 1 by -1 do
        for each arc (i,j) in G such that i<j do
            REVISE((i,j))
            if $D_i = \emptyset$ then stop with fail
        end for
    end for
end DAC-1
```



How to exploit DAC?

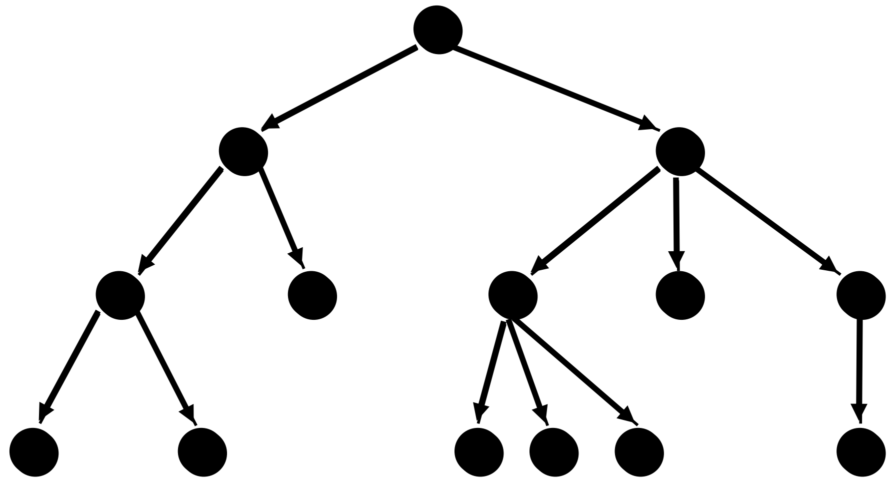
Obviously AC covers DAC (if CSP is AC, then it is DAC).

Is DAC anyhow useful?

- DAC-1 is clearly more efficient than AC-x
- Moreover, there are problems where DAC is enough.

Example: If the constraint network is a tree then we can use DAC to solve the problem in a backtrack-free way.

- **How to order the nodes for DAC?**
- **How to order the nodes for labelling (search)?**



1. Apply DAC in the order of nodes from the root.

2. Label (assign) the nodes starting at root.

DAC ensures that for each child node there is a value compatible with the parent node.

Relation between DAC and AC

Observation:

A CSP is arc consistent if for some ordering of variables the problem is directional arc consistent in both directions (according to that ordering).

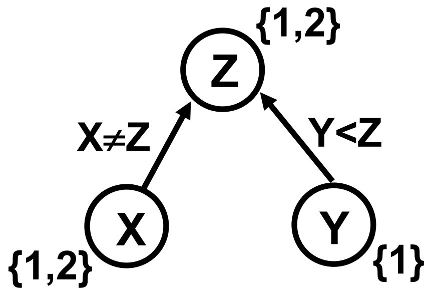
Can we make the problem AC by applying DAC in both directions?

In general NO, but…

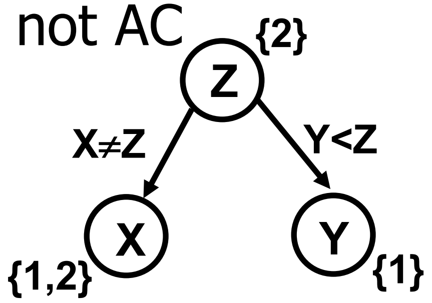
Example:

X in {1,2}, Y in {1}, Z in {1,2}, X≠Z,Y<Z

for ordering X,Y,Z there is no change in domains



for ordering Z,Y,X the domain of Z is pruned, but the problem is not AC



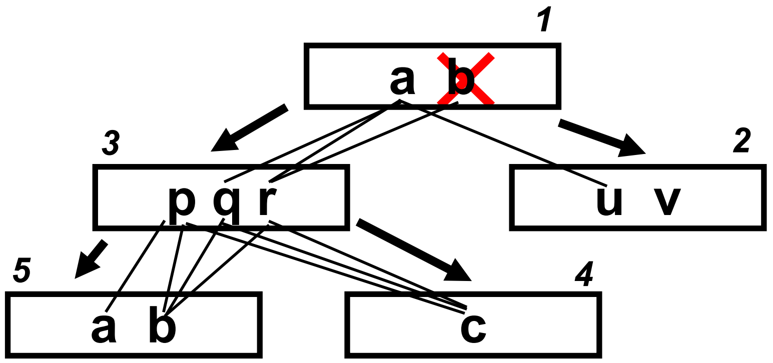
If we first try the ordering Z,Y,X, then we get AC!

From DAC to AC for tree-structured CSPs

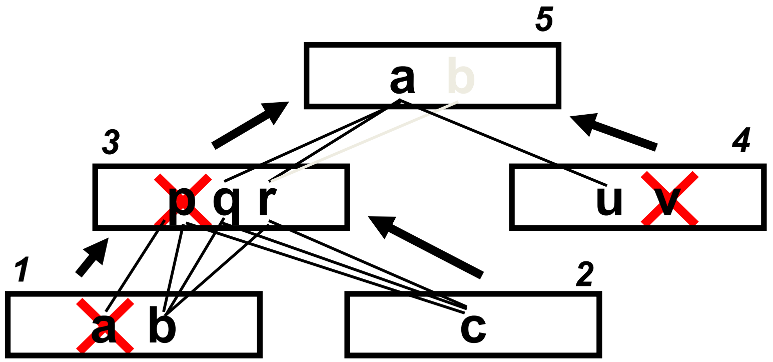
If we apply DAC to a tree-structured CSP first for the ordering from the root and then in the reverse direction from leafs then we obtain AC.

Proof:

after the first run of DAC we ensure that each value in a parent node has a support (a compatible value) in all child nodes



if some value is deleted during the **second run of DAC** (in the reverse order) then this value is not a support for any value in the parent node (so the values in the parent do not lose supports)



together: each value has a support in all child nodes (the first DAC run) and in the parent node too (the second DAC run) so the value is AC

Non-binary constraints

So far we assumed mainly **binary constraints**.

We can use binary constraints, because **every CSP can be converted to a binary CSP**!

Is this really done in practice?

- in many applications, non-binary constraints are naturally used, for example, $a+b+c \leq 5$
- for such constraints we can do some local inference / propagation

  for example, if we know that $a,b \geq 2$, we can deduce that $c \leq 1$
- Within a single constraint, we can restrict the domains of variables to the values satisfying the constraint

  ⇘ **generalized arc consistency**

Generalized arc consistency

- **The value** x of variable V is **generalized arc consistent** with respect to constraint P if and only if there exist values for the other variables in P such that together with x they satisfy the constraint P

  Example: A+B≤C, A in {1,2,3}, B in {1,2,3}, C in {1,2,3}
  Value 1 for C is not GAC (it has no support), 2 and 3 are GAC.

- **The variable** V is **generalized arc consistent** with respect to constraint P, if and only if all values from the current domain of V are GAC with respect to P.

  Example: A+B≤C, A in {1,2,3}, B in {1,2,3}, C in {2,3}
  C is GAC, A and B are not GAC

- **The constraint** C is **generalized arc consistent**, if and only if all variables in C are GAC.

  Example: for A in {1,2}, B in {1,2}, C in {2,3} A+B≤C is GAC

- **The constraint satisfaction problem** P is **generalized arc consistent**, if and only if all the constraints in P are GAC.

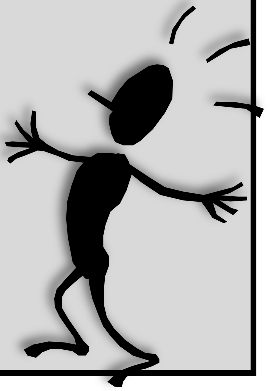
How to make a CSP GAC?

We will modify AC-3 for non-binary constraints.

- We can see a constraint as a set of propagation methods – each method makes one variable GAC:

  $A + B = C$: $A + B \rightarrow C$, $C - A \rightarrow B$, $C - B \rightarrow A$
- By executing all the methods we make the constraint GAC.
- We repeat revisions until any domain changes.

```
procedure GAC-3(G)
    Q ← {Xs →Y | Xs →Y is a method for some constraint in G}
    while Q non empty do
        select and delete (As→B) from Q
        if REVISE(As→B) then
            if $D_B=\emptyset$ then stop with fail
            Q ← Q ∪ {Xs →Y | Xs →Y is a method s.t. $B\in Xs$}
        end if
    end while
end GAC-3
```



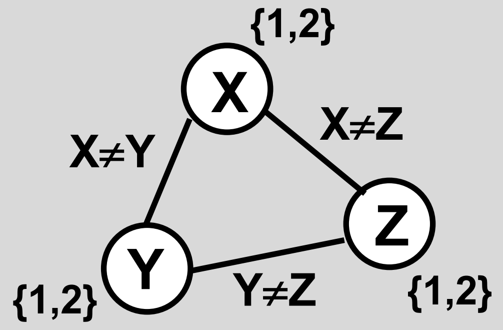
Is AC enough to solve CSPs?

By applying AC we remove many inconsistent values.

– **Did we solve the problem?**
– **Do we know that a solution exists?**

NO and NO!

Example:



CSP is arc consistent
but there is no solution!

What is advantage of using AC?

– Sometimes **AC directly provides a solution**.
  - any domain is empty → no solution exists
  - all domains are singleton → this is a solution
– In general, AC **decreases the search space.**

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