Logic-Based Explainable Artificial Intelligence

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ESSAI, Athens, Greece, July 2024

My team's recent & not so recent work...

- SAT Solving (Clause learning, UIPs, ...)
- Quantification & CEGAR (QBF, QMaxSAT, etc.)
- Function Synthesis (Min DNF cover, ...)
- Inconsistency (MUS, MCS, etc.)
- Certification of Reasoners
- Model Checking, Synthesizing Invariants, ATPG, Reconfiguration
- Optimization (MaxSAT, MinSAT, PBO, WBO, etc.)
- Propositional Encodings, Backbones, Autarkies, Minimal models, etc.
- Enumeration (MUSes, MCSes, etc.)
- Proof Systems (DRMaxSAT, etc.)
- Primes, Abduction, DLs, etc.

New area of research, since circa 2018...

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- Explainability & Interpretability in ML

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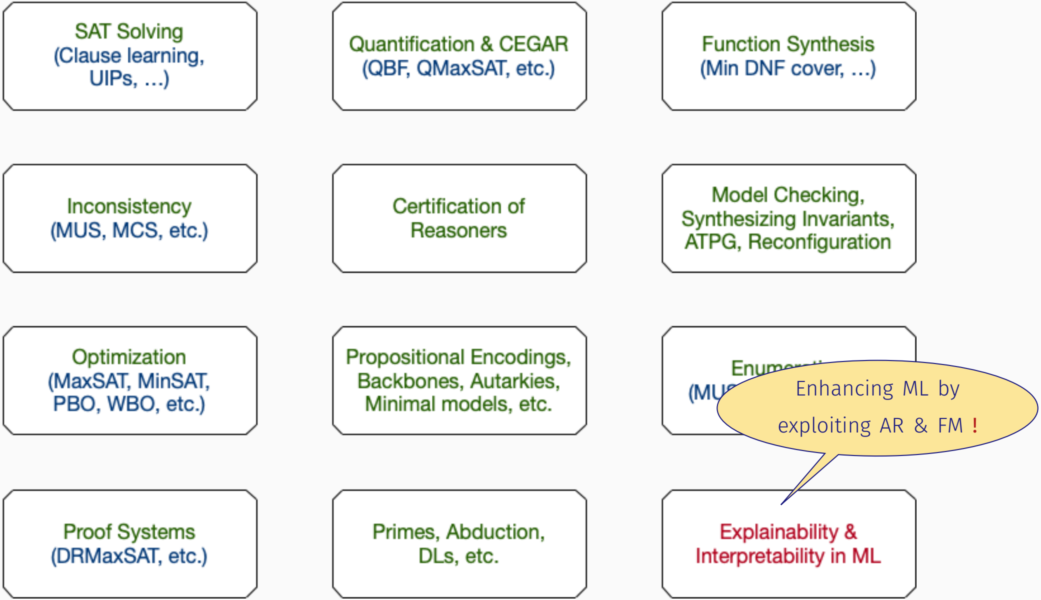


Lecture 01

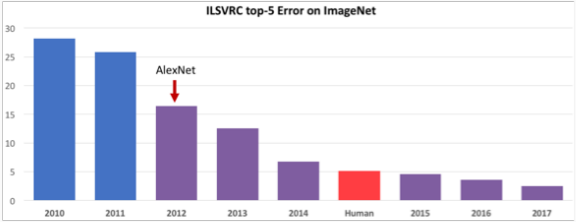
https://en.wikipedia.org/wiki/Waymo

DeepMind

AlphaGo

AlphaGo Zero & **Alpha Zero**

Image & Speech Recognition



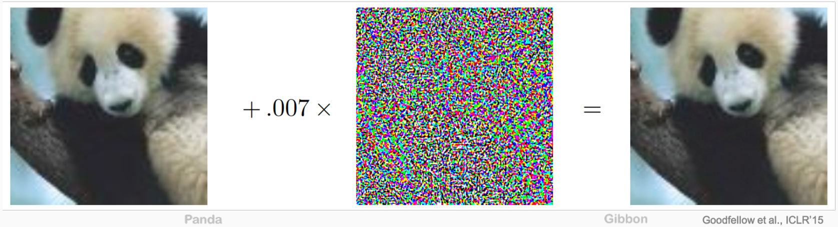
http://gradientscience.org/intro_adversarial/

https://fr.wikipedia.org/wiki/Pepper_(robot)

Can we trust ML models?

- Accuracy in training/test data
- Complex ML models are brittle
  - Extensive work on finding adversarial examples
  - Extensive work on learning robust ML models
- More recently, complex ML models hallucinate
- One **must** be able to validate operation of ML model, with rigor
  - Explanations; robustness; verification

ML models are brittle — adversarial examples



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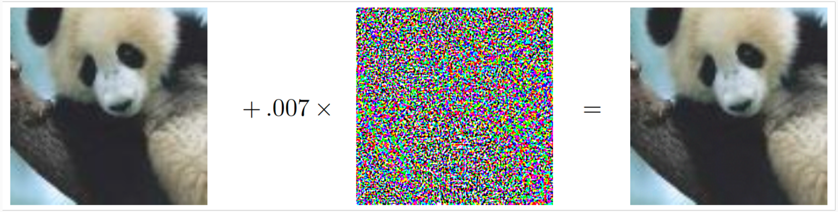
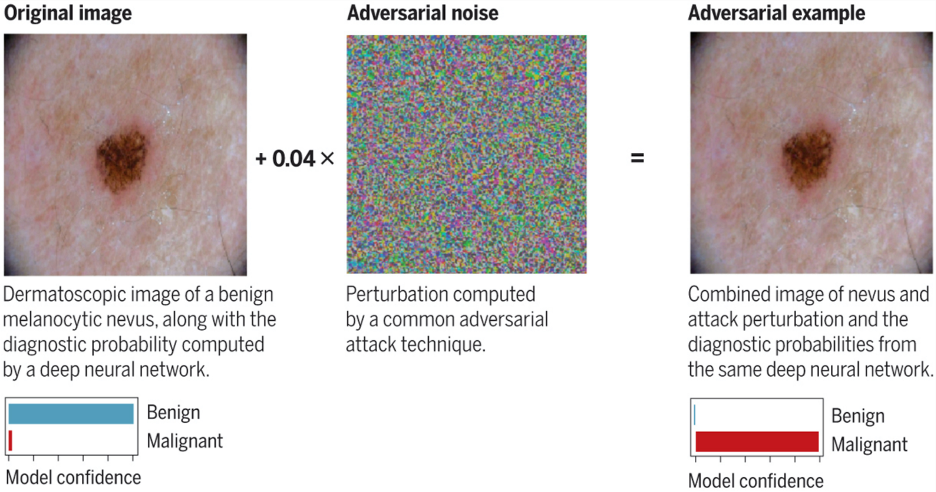


Panda Gibbon Goodfellow et al., ICLR'15

Eykholt et al'18 Aung et al'17

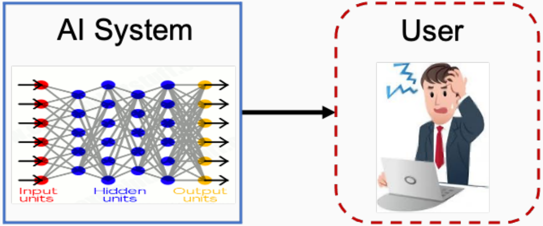
ML models are brittle — adversarial examples

Adversarial examples can be very problematic



Finlayson et al., Nature 2019

eXplainable AI (XAI)



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- Goal of XAI: **to help humans understand ML models**
- Many questions to address:

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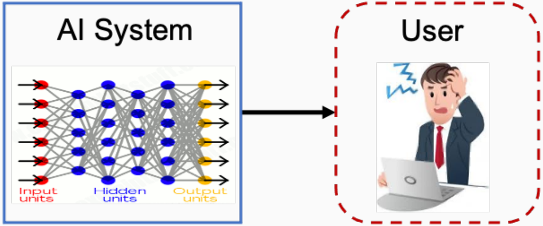

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    - How to be human understandable?
    - How to answer **Why?** questions? I.e. Why the prediction?
    - How to answer **Why Not?** questions? I.e. Why not some other prediction?
    - Which guarantees of rigor?

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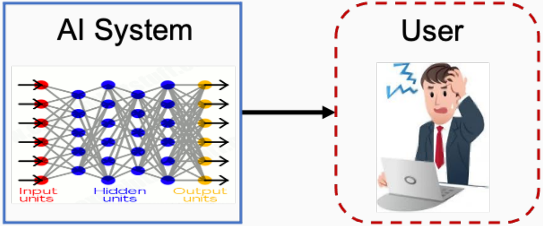

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    - Which guarantees of rigor?
  - Other queries: enumeration, membership, preferences, etc.
  - Links with robustness, fairness, model learning

REGULATION (EU) 2016/679 OF THE EUROPEAN PARLIAMENT AND OF THE COUNCIL

of 27 April 2016

on the protection of natural persons with regard to the processing of personal data and on the free movement of such data, and repealing Directive 95/46/EC (General Data Protection Regulation)

(Text with EEA relevance)

European Union regulations on algorithmic decision-making and a "right to explanation"

Bryce Goodman,[1*] Seth Flaxman,[2]

Proposal for a

REGULATION OF THE EUROPEAN PARLIAMENT AND OF THE COUNCIL

LAYING DOWN HARMONISED RULES ON ARTIFICIAL INTELLIGENCE (ARTIFICIAL INTELLIGENCE ACT) AND AMENDING CERTAIN UNION LEGISLATIVE ACTS

■ *We summarize the potential impact that the European Union's new General Data Protection Regulation will have on the routine use of machine-learning algorithms. Slated to take effect as law across the European Union in 2018, it will place restrictions on automated individual decision making (that is, algorithms that make decisions based on user-level predictors) that "significantly affect" users. When put into practice, the law may also effectively create a right to explanation, whereby a user can ask for an explanation of an algorithmic decision that significantly affects them. We argue that while this law may pose large challenges for industry, it highlights opportunities for computer scientists to take the lead in designing algorithms and evaluation frameworks that avoid discrimination and enable explanation.*

Explainable Artificial Intelligence (XAI)

XAI
EXPLAINABLE ARTIFICIAL INTELLIGENCE

FY17 FY18 FY19 FY20 FY21

David Gunning
DARPA/I2O
Program Update November 2017

DARPA

©DARPA

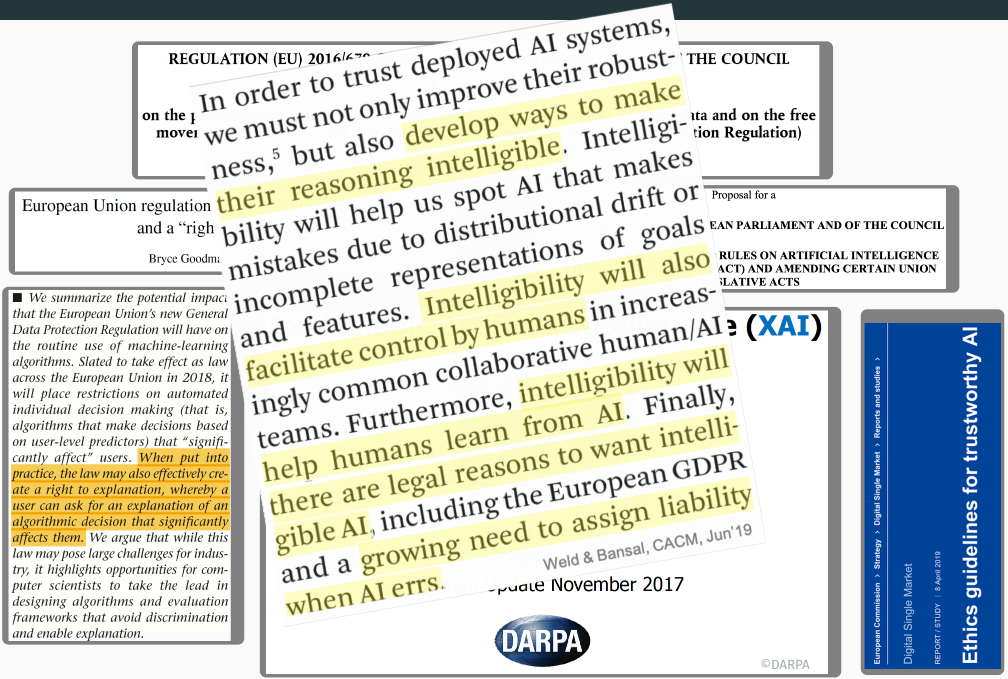
European Commission > Strategy > Digital Single Market > Reports and studies >

Digital Single Market

REPORT / STUDY | 8 April 2019

Ethics guidelines for trustworthy AI

Importance of XAI



XAI & EU guidelines (AI HLEG)

Search

European Commission › Strategy › Digital Single Market › Reports and studies ›

Digital Single Market

REPORT / STUDY | 8 April 2019

Ethics guidelines for trustworthy AI

Following the publication of the draft ethics guidelines in December 2018 to which more than 500 comments were received, the independent expert group presents today their ethics guidelines for trustworthy artificial intelligence.

About Artificial intelligence

Blog posts

News

XAI & the principle of explicability

European Commission › Strategy › Digital Single Market › Reports and

Digital Single Market

REPORT / ST

- *The principle of explicability*

Explicability is crucial for building and maintaining users' trust in AI systems. This means that processes need to be transparent, the capabilities and purpose of AI systems openly communicated, and decisions – to the extent possible – explainable to those directly and indirectly affected. Without such information, a decision cannot be duly contested. An explanation as to why a model has generated a particular output or decision (and what combination of input factors contributed to that) is not always possible. These cases are referred to as 'black box' algorithms and require special attention. In those circumstances, other explicability measures (e.g. traceability, auditability and transparent communication on system capabilities) may be required, provided that the system as a whole respects fundamental rights. The degree to which explicability is needed is highly dependent on the context and the severity of the consequences if that output is erroneous or otherwise inaccurate.[33]

...ents were ... group presents today their ... or trustworthy artificial intelligence.

About Artificial intelligence

Blog posts

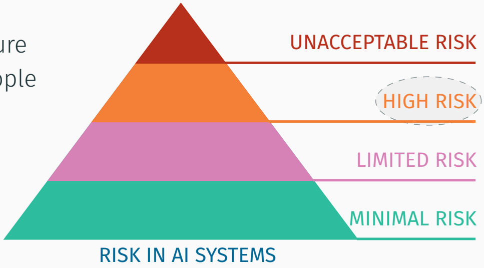
News

& thousands of recent papers!

XAI for high-risk & safety-critical applications

- **High-risk** (EU regulations):
  - Law enforcement
  - Management and operation of critical infrastructure
  - Biometric identification and categorization of people
  - ...

[EU21b, EU21a]



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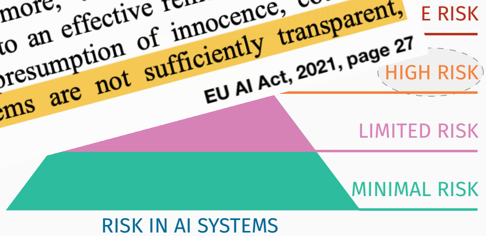
21b, EU21a]

otherwise incorrect or unjust manner. Furthermore, the exercise of important procedural fundamental rights, such as the right to an effective remedy and to a fair trial as well as the right of defence and the presumption of innocence, could be hampered, in particular, where such AI systems are not sufficiently transparent, explainable and documented.

EU AI Act, 2021, page 27



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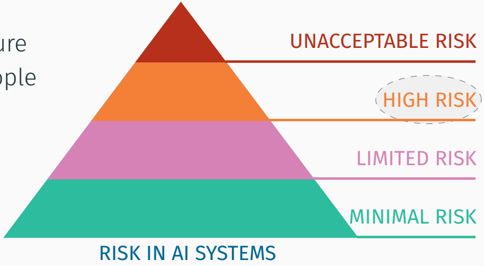

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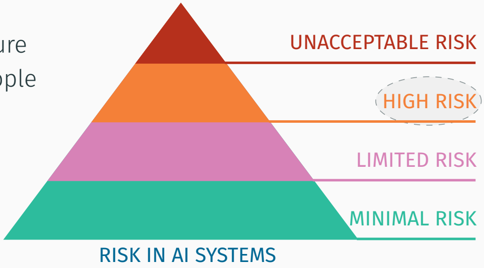

PERSPECTIVE
https://doi.org/10.1038/s42256-019-0048-x
nature machine intelligence

Stop explaining black box machine learning models for high stakes decisions and use interpretable models instead

Cynthia Rudin — May 2019

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  - To build trust
  - To help debug AI systems
  - To prevent (catastrophic) accidents
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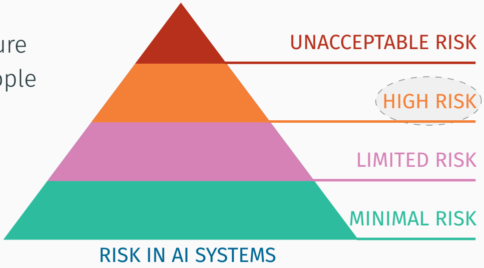
[EU21b, EU21a]

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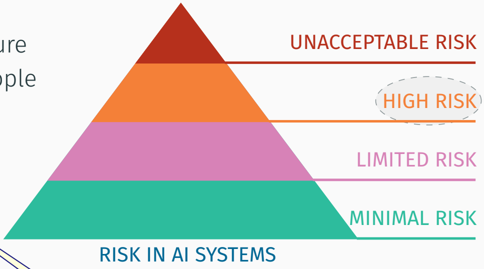

Main motivation for our work! (since 2019)

Can we trust (non-symbolic) XAI? – some questions

- Many proposed **solutions** for XAI
  - Most, and the better-known, are heuristic
  - I.e. no guarantees of rigor
- Many proposed **uses** of XAI
- Regular complaints about issues with existing (heuristic) methods of XAI

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- Many proposed **uses** of XAI
- Regular complaints about issues with existing (heuristic) methods of XAI

- **Q:** Can heuristic XAI be trusted in high-risk and/or safety-critical domains?
- **Q:** Can we validate results of heuristic XAI?

What have we been up to? 1. Created the field of symbolic (formal) XAI – I

[MI22, Mar22, MS23, Mar24]

- Rigorous, logic-based, definitions of explanations
  - Relationship with abduction – abductive explanations (AXps)
  - Contrastive explanations (CXps) [Mil19]
- Duality between AXps & CXps
  - AXps are MHSes of CXps and vice-versa

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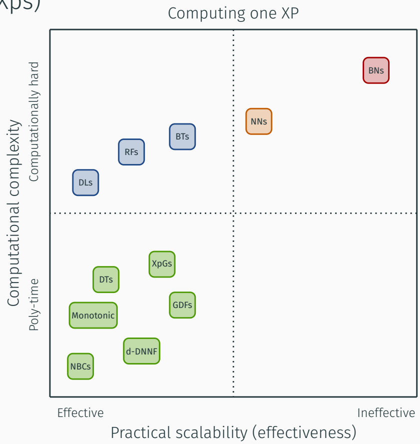
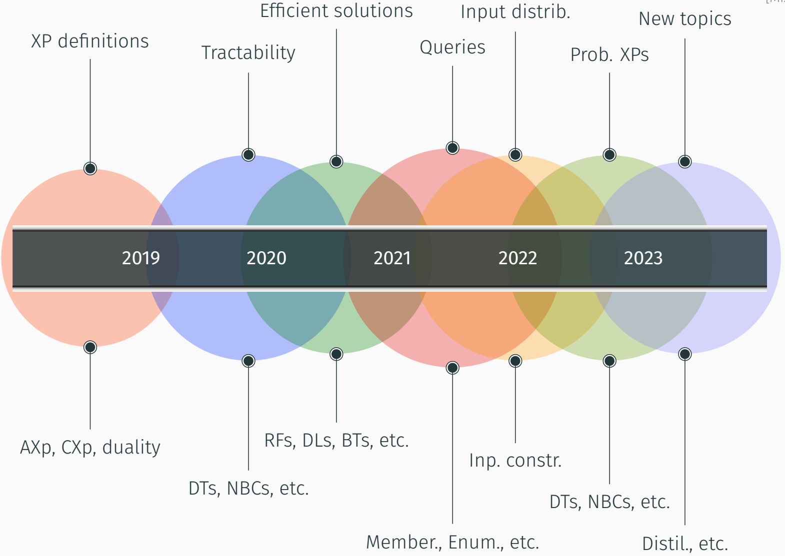


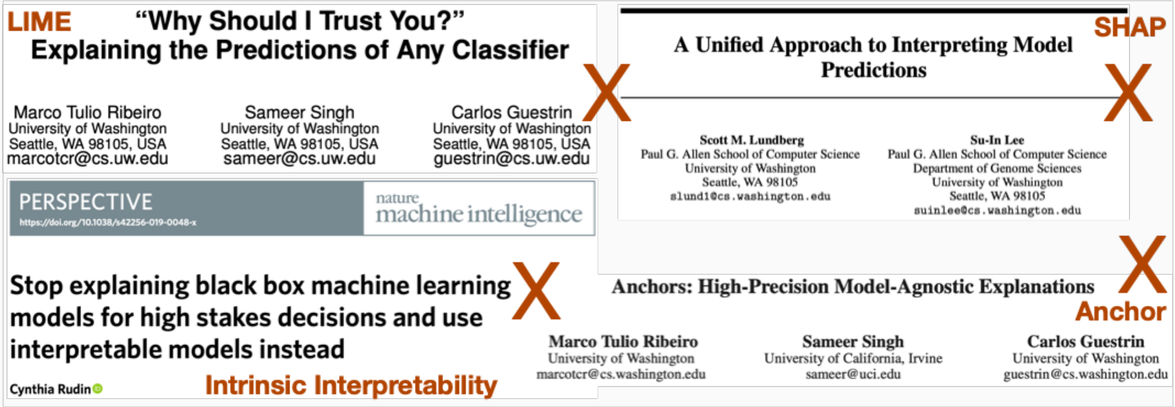
What have we been up to? 1. Created the field of symbolic (formal) XAI – II

[MI22, Mar22, MS23, Mar24]



What have we been up to? 2. Uncovered key myths of non-symbolic XAI – I

[RSG16, LL17, RSG18, Rud19]



[MSH24, HMS24, HM23c]

research and advances

DOI:10.1145/3635301

When the decisions of ML models impact people, one should expect explanations to offer the strongest guarantees of rigor. However, the most popular XAI approaches offer none.

BY JOAO MARQUES-SILVA AND XUANXIANG HUANG

Explainability Is *Not* a Game

key insights

- Shapley values find extensive uses in explaining machine learning models and serve to assign importance to the features of the model.
- Shapley values for explainability also find ever-increasing uses in high-risk and safety-critical domains, for example, medical diagnosis.
- This article proves that the existing definition of Shapley values for explainability can produce misleading information regarding feature importance, and so can induce human decision makers in error.

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Plan for this course

- Lecture 01 – units:
  - #01: Foundations
- Lecture 02 – units:
  - #02: Principles of symbolic XAI – feature selection
  - #03: Tractability in symbolic XAI (& myth of interpretability)
- Lecture 03 – units:
  - #04: Intractability in symbolic XAI (& myth of model-agnostic XAI)
  - #05: Explainability queries
- Lecture 04 – units:
  - #06: Advanced topics
- Lecture 05 – units:
  - #07: Principles of symbolic XAI – feature attribution (& myth of Shapley values in XAI)
  - #08: Conclusions & research directions

Unit #01

Foundations

Classification problems

- Set of features $\mathcal{F} = \{1, 2, \ldots, m\}$, each feature i taking values from domain D_i
  - Features can be categorical, discrete or real-valued
  - Feature space: $\mathbb{F} = \Pi_{i=1}^{m} D_i$

- Set of classes $\mathcal{K} = \{c_1, \ldots, c_K\}$

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  - $\mathcal{M}_C$ is a tuple $(\mathcal{F}, \mathbb{F}, \mathcal{K}, \kappa)$
- Instance $(\mathbf{v}, c)$ for point $\mathbf{v} = (v_1, \ldots, v_m) \in \mathbb{F}$, with prediction $c = \kappa(\mathbf{v})$, $c \in \mathcal{K}$
  - **Goal:** to compute explanations for $(\mathbf{v}, c)$

Regression problems

- For regression problems:
  - Codomain: $\mathbb{V}$
  - Regression function: $\rho : \mathbb{F} \to \mathbb{V}$ (non-constant)
  - ML model: $\mathcal{M}_R$ is a tuple $(\mathcal{F}, \mathbb{F}, \mathbb{V}, \rho)$

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- General ML model:
  - $\mathbb{T}$: range of possible predictions
  - Non-constant function $\tau : \mathbb{F} \to \mathbb{T}$
  - ML model: $\mathcal{M}$ is a tuple $(\mathcal{F}, \mathbb{F}, \mathbb{T}, \tau)$

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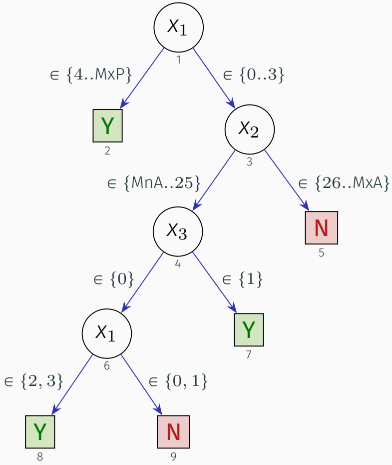
- Instance: $(\mathbf{v}, q)$, $q \in \mathbb{T}$

Example ML models – classification – decision trees (DTs)



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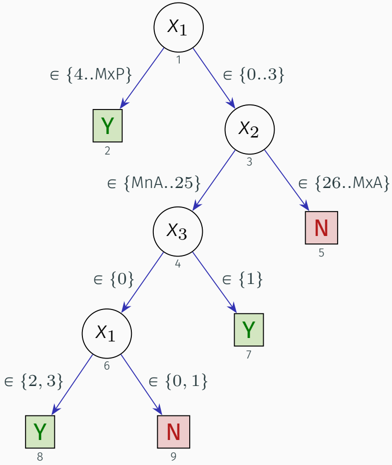
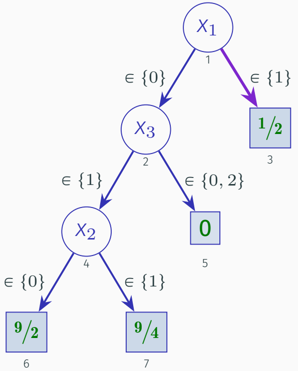


- Literals in DTs can use $=$ or $\in$

Example ML models – regression – regression trees (RTs)



- Literals in RTs can use $=$ or $\in$

- Ordered rules – decision lists (DLs):

IF	$x_1 \land x_2$	THEN	predict Y
ELSE IF	$\neg x_2 \lor x_3$	THEN	predict N
ELSE		THEN	predict Y

$\mathcal{F} = \{1, 2, 3\}; \mathcal{D}_1 = \mathcal{D}_2 = \mathcal{D}_3 = \{0, 1\}; \mathcal{K} = \{\mathsf{Y}, \mathsf{N}\}$

Example ML models – classification – rules

- Ordered rules – decision lists (DLs):

$$
\begin{array}{llll}
\text{IF} & x_1 \land x_2 & \text{THEN} & \text{predict } \mathsf{Y} \\
\text{ELSE IF} & \neg x_2 \lor x_3 & \text{THEN} & \text{predict } \mathsf{N} \\
\text{ELSE} & & \text{THEN} & \text{predict } \mathsf{Y}
\end{array}
$$

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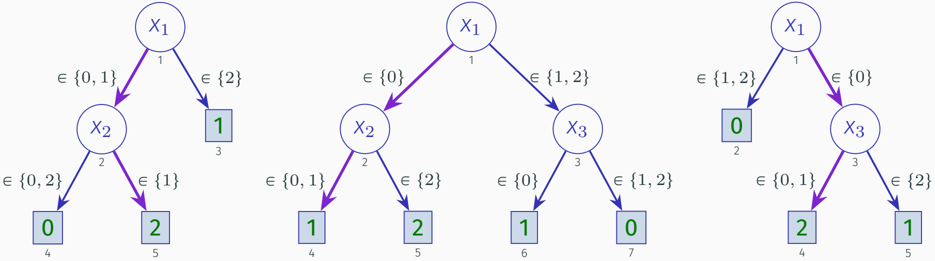
- Unordered rules – decision sets (DSs):

$$
\begin{array}{llll}
\text{IF} & x_1 + x_2 \geqslant 0 & \text{THEN} & \text{predict } \boxplus \\
\text{IF} & x_1 + x_2 < 0 & \text{THEN} & \text{predict } \boxminus
\end{array}
$$

$$\mathcal{F} = \{1,2\}; \mathcal{D}_1 = \mathcal{D}_2 = \mathbb{R}; \mathcal{K} = \{\boxplus, \boxminus\}$$

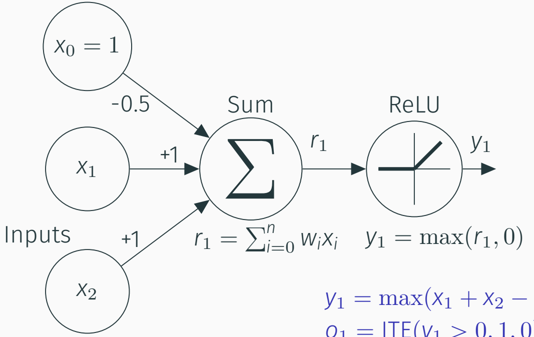
  - Issues of DSs: overlap; incomplete coverage

Example ML models – classification – random forests (RFs)



- For each input, each DT picks a class
- Result uses majority or weighted voting of the DTs

Example ML models – classification – neural networks (NNs)



$$y_1 = \max(x_1 + x_2 - 0.5, 0)$$
$$o_1 = \text{ITE}(y_1 > 0, 1, 0)$$

Outline – Unit #01

ML Models: Classification & Regression Problems

Basics of (non-symbolic) XAI

Motivation for Explanations

Brief Glimpse of Logic

Reasoning About ML Models

Understanding Intrinsic Interpretability

Basics of (non-symbolic) XAI – more detail later

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  - LIME [RSG16]
  - SHAP [LL17]
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Some examples

- Anchors: [RSG18]

IF Country = United-States **AND** Capital Loss = Low
AND Race = White **AND** Relationship = Husband
AND Married **AND** $28 < \text{Age} \leq 37$
AND Sex = Male **AND** High School grad
AND Occupation = Blue-Collar
THEN PREDICT Salary > \$50K

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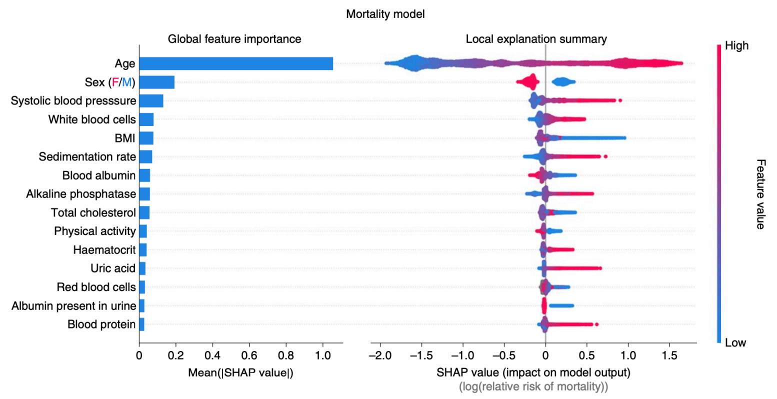
- SHAP: [LL17, LEC+20]



Outline – Unit #01

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    - <COND> is irreducible
  - We also seek the algorithms for the rigorous computation of such rules

A decision list example

IF	$\neg x_1 \wedge x_2$	THEN	predict Y
ELSE IF	$\neg x_1 \wedge x_3$	THEN	predict Y
ELSE IF	$x_4 \wedge x_5$	THEN	predict N
ELSE		THEN	predict Y

A decision list example

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  - Given $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5)$,
    IF $(x_1 = 1) \land (x_4 = 1) \land (x_5 = 1)$ **THEN** $\kappa(\mathbf{x}) = \mathsf{N}$
  - I.e. $\{x_1 = 1, x_4 = 1, x_5 = 1\}$ suffice for DL to predict **N**

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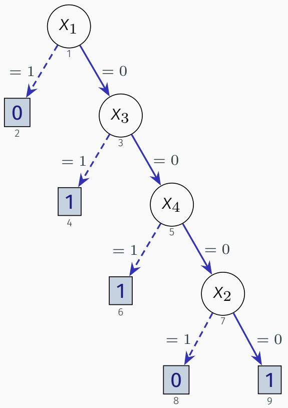
    IF $(x_4 = 0)$ THEN $\kappa(\mathbf{x}) = \text{Y}$
  - I.e. $\{x_4 = 0\}$ suffices for DL to predict Y
  - Given $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5)$,
    IF $(x_5 = 0)$ THEN $\kappa(\mathbf{x}) = \text{Y}$
  - I.e. $\{x_5 = 0\}$ also suffices for DL to predict Y

A decision tree example



x_1	x_2	x_3	x_4	$\kappa(\mathbf{x})$
0	0	0	0	1
0	0	0	1	1
0	0	1	0	1
0	0	1	1	1
0	1	0	0	0
0	1	0	1	1
0	1	1	0	1
0	1	1	1	1
1	0	0	0	0
1	0	0	1	0
1	0	1	0	0
1	0	1	1	0
1	1	0	0	0
1	1	0	1	0
1	1	1	0	0
1	1	1	1	0

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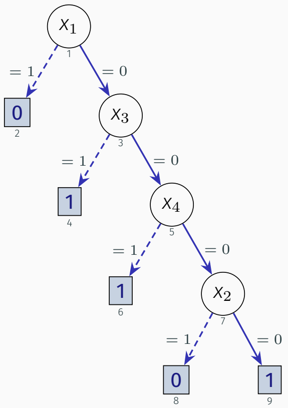

- Explanation for why $\kappa(0, 0, 0, 0) = 1$?

x_1	x_2	x_3	x_4	$\kappa(\mathbf{x})$
0	0	0	0	1
0	0	0	1	1
0	0	1	0	1
0	0	1	1	1
0	1	0	0	0
0	1	0	1	1
0	1	1	0	1
0	1	1	1	1
1	0	0	0	0
1	0	0	1	0
1	0	1	0	0
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1	1	0	0	0
1	1	0	1	0
1	1	1	0	0
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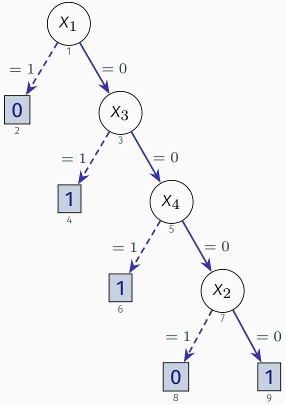

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  - Given $\mathbf{x} = (x_1, x_2, x_3, x_4)$,
    IF $(x_1 = 0) \wedge (x_2 = 0)$ THEN $\kappa(\mathbf{x}) = 1$
  - I.e. $\{x_1 = 0, x_2 = 0\}$ suffice for DT to predict **1**

x_1	x_2	x_3	x_4	$\kappa(\mathbf{x})$
0	0	0	0	1
0	0	0	1	1
0	0	1	0	1
0	0	1	1	1
0	1	0	0	0
0	1	0	1	1
0	1	1	0	1
0	1	1	1	1
1	0	0	0	0
1	0	0	1	0
1	0	1	0	0
1	0	1	1	0
1	1	0	0	0
1	1	0	1	0
1	1	1	0	0
1	1	1	1	0

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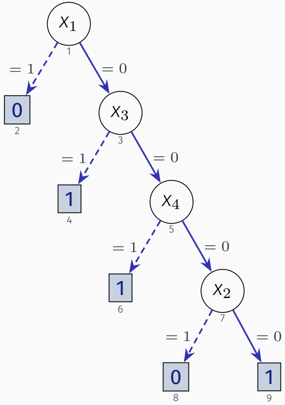

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  - I.e. $\{x_1 = 0, x_2 = 0\}$ suffice for DT to predict 1
- Explanation for why $\kappa(1,1,1,1) = 0$?

x_1	x_2	x_3	x_4	$\kappa(\mathbf{x})$
0	0	0	0	1
0	0	0	1	1
0	0	1	0	1
0	0	1	1	1
0	1	0	0	0
0	1	0	1	1
0	1	1	0	1
0	1	1	1	1
1	0	0	0	0
1	0	0	1	0
1	0	1	0	0
1	0	1	1	0
1	1	0	0	0
1	1	0	1	0
1	1	1	0	0
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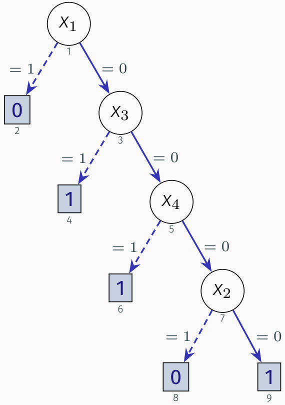

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    IF $(x_1 = 1)$ THEN $\kappa(\mathbf{x}) = 0$
  - I.e. $\{x_1 = 1\}$ suffices for DT to predict 0

x_1	x_2	x_3	x_4	$\kappa(\mathbf{x})$
0	0	0	0	1
0	0	0	1	1
0	0	1	0	1
0	0	1	1	1
0	1	0	0	0
0	1	0	1	1
0	1	1	0	1
0	1	1	1	1
1	0	0	0	0
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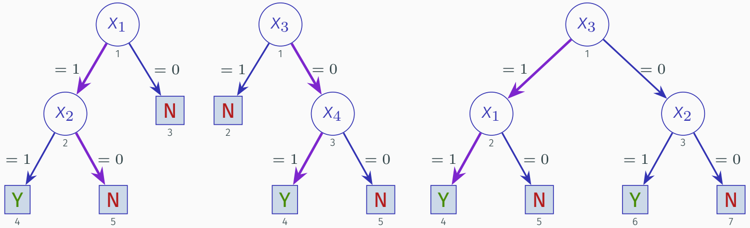
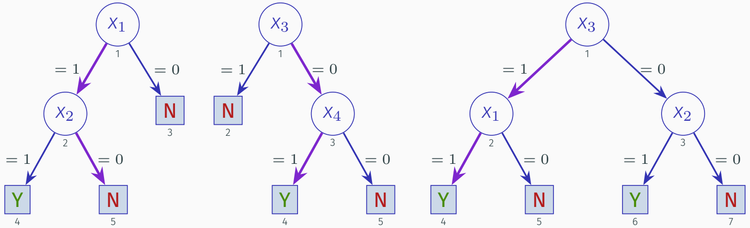
1	0	1	1	0
1	1	0	0	0
1	1	0	1	0
1	1	1	0	0
1	1	1	1	0

A random forest example

[IMS21]



x_1	x_2	x_3	x_4	T_1	T_2	T_3	$\kappa(\mathbf{x})$
0	0	0	0	N	N	N	N
0	0	0	1	N	Y	N	N
0	0	1	0	N	N	N	N
0	0	1	1	N	N	N	N
0	1	0	0	N	N	Y	N
0	1	0	1	N	Y	Y	Y
0	1	1	0	N	N	N	N
0	1	1	1	N	N	N	N
1	0	0	0	N	N	N	N
1	0	0	1	N	Y	N	N
1	0	1	0	N	N	Y	N
1	0	1	1	N	N	Y	N
1	1	0	0	Y	N	Y	Y
1	1	0	1	Y	Y	Y	Y
1	1	1	0	Y	N	Y	Y
1	1	1	1	Y	N	Y	Y



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x_1	x_2	x_3	x_4	T_1	T_2	T_3	$\kappa(\mathbf{x})$
0	0	0	0	N	N	N	N
0	0	0	1	N	Y	N	N
0	0	1	0	N	N	N	N
0	0	1	1	N	N	N	N
0	1	0	0	N	N	Y	N
0	1	0	1	N	Y	Y	Y
0	1	1	0	N	N	N	N
0	1	1	1	N	N	N	N
1	0	0	0	N	N	N	N
1	0	0	1	N	Y	N	N
1	0	1	0	N	N	Y	N
1	0	1	1	N	N	Y	N
1	1	0	0	Y	N	Y	Y
1	1	0	1	Y	Y	Y	Y
1	1	1	0	Y	N	Y	Y
1	1	1	1	Y	N	Y	Y

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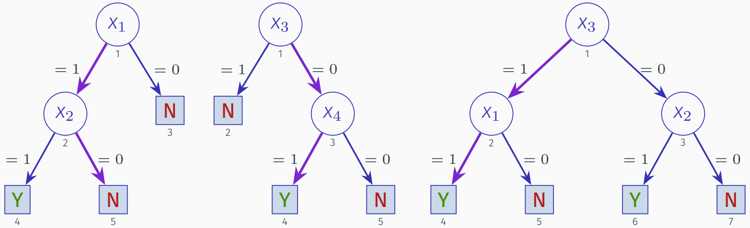

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0	1	0	0	N	N	Y	N
0	1	0	1	N	Y	Y	Y
0	1	1	0	N	N	N	N
0	1	1	1	N	N	N	N
1	0	0	0	N	N	N	N
1	0	0	1	N	Y	N	N
1	0	1	0	N	N	Y	N
1	0	1	1	N	N	Y	N
1	1	0	0	Y	N	Y	Y
1	1	0	1	Y	Y	Y	Y
1	1	1	0	Y	N	Y	Y
1	1	1	1	Y	N	Y	Y

- Explanation for why $\kappa(1,0,0,1) = \text{N}$?
  - Given $\mathbf{x} = (x_1, x_2, x_3, x_4)$, IF $(x_2 = 0)$ THEN $\kappa(\mathbf{x}) = \text{N}$
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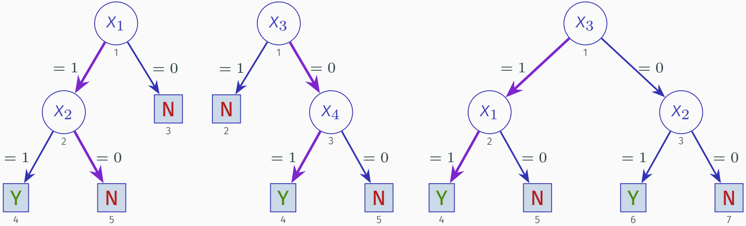

x_1	x_2	x_3	x_4	T_1	T_2	T_3	$\kappa(\mathbf{x})$
0	0	0	0	N	N	N	N
0	0	0	1	N	Y	N	N
0	0	1	0	N	N	N	N
0	0	1	1	N	N	N	N
0	1	0	0	N	N	Y	N
0	1	0	1	N	Y	Y	Y
0	1	1	0	N	N	N	N
0	1	1	1	N	N	N	N
1	0	0	0	N	N	N	N
1	0	0	1	N	Y	N	N
1	0	1	0	N	N	Y	N
1	0	1	1	N	N	Y	N
1	1	0	0	Y	N	Y	Y
1	1	0	1	Y	Y	Y	Y
1	1	1	0	Y	N	Y	Y
1	1	1	1	Y	N	Y	Y

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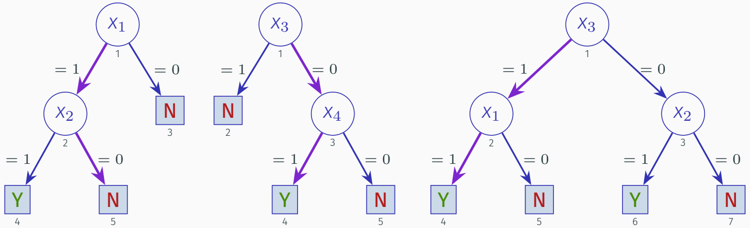

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x_1	x_2	x_3	x_4	T_1	T_2	T_3	$\kappa(\mathbf{x})$
0	0	0	0	N	N	N	N
0	0	0	1	N	Y	N	N
0	0	1	0	N	N	N	N
0	0	1	1	N	N	N	N
0	1	0	0	N	N	Y	N
0	1	0	1	N	Y	Y	Y
0	1	1	0	N	N	N	N
0	1	1	1	N	N	N	N
1	0	0	0	N	N	N	N
1	0	0	1	N	Y	N	N
1	0	1	0	N	N	Y	N
1	0	1	1	N	N	Y	N
1	1	0	0	Y	N	Y	Y
1	1	0	1	Y	Y	Y	Y
1	1	1	0	Y	N	Y	Y
1	1	1	1	Y	N	Y	Y

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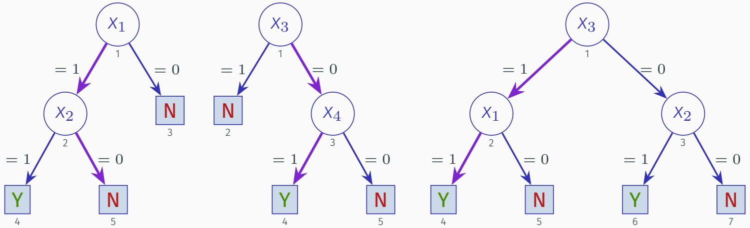
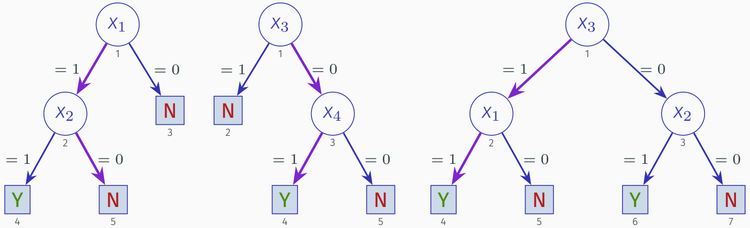
[IMS21]



x_1	x_2	x_3	x_4	T_1	T_2	T_3	$\kappa(\mathbf{x})$
0	0	0	0	N	N	N	N
0	0	0	1	N	Y	N	N
0	0	1	0	N	N	N	N
0	0	1	1	N	N	N	N
0	1	0	0	N	N	Y	N
0	1	0	1	N	Y	Y	Y
0	1	1	0	N	N	N	N
0	1	1	1	N	N	N	N
1	0	0	0	N	N	N	N
1	0	0	1	N	Y	N	N
1	0	1	0	N	N	Y	N
1	0	1	1	N	N	Y	N
1	1	0	0	Y	N	Y	Y
1	1	0	1	Y	Y	Y	Y
1	1	1	0	Y	N	Y	Y
1	1	1	1	Y	N	Y	Y

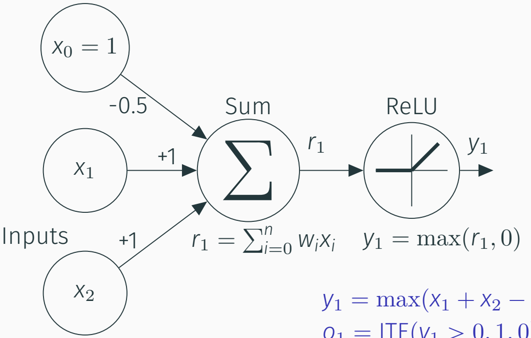
- Explanation for why $\kappa(1,0,0,1) = \text{N}$?
  - Given $\mathbf{x} = (x_1, x_2, x_3, x_4)$, IF $(x_2 = 0)$ THEN $\kappa(\mathbf{x}) = \text{N}$
  - I.e. $\{x_2 = 0\}$ suffices for DT to predict N
- Explanation for why $\kappa(1,1,1,1) = \text{Y}$?
  - Given $\mathbf{x} = (x_1, x_2, x_3, x_4)$, IF $(x_1 = 1) \wedge (x_2 = 1)$ THEN $\kappa(\mathbf{x}) = \text{Y}$
  - I.e. $\{x_1 = 1, x_2 = 1\}$ suffice for DT to predict Y
- Explanation for why $\kappa(0,1,1,1) = \text{N}$?



x_1	x_2	x_3	x_4	T_1	T_2	T_3	$\kappa(\mathbf{x})$
0	0	0	0	N	N	N	N
0	0	0	1	N	Y	N	N
0	0	1	0	N	N	N	N
0	0	1	1	N	N	N	N
0	1	0	0	N	N	Y	N
0	1	0	1	N	Y	Y	Y
0	1	1	0	N	N	N	N
0	1	1	1	N	N	N	N
1	0	0	0	N	N	N	N
1	0	0	1	N	Y	N	N
1	0	1	0	N	N	Y	N
1	0	1	1	N	N	Y	N
1	1	0	0	Y	N	Y	Y
1	1	0	1	Y	Y	Y	Y
1	1	1	0	Y	N	Y	Y
1	1	1	1	Y	N	Y	Y

- Explanation for why $\kappa(1,0,0,1) = \text{N}$?
  - Given $\mathbf{x} = (x_1, x_2, x_3, x_4)$, IF $(x_2 = 0)$ THEN $\kappa(\mathbf{x}) = \text{N}$
  - I.e. $\{x_2 = 0\}$ suffices for DT to predict N
- Explanation for why $\kappa(1,1,1,1) = \text{Y}$?
  - Given $\mathbf{x} = (x_1, x_2, x_3, x_4)$, IF $(x_1 = 1) \wedge (x_2 = 1)$ THEN $\kappa(\mathbf{x}) = \text{Y}$
  - I.e. $\{x_1 = 1, x_2 = 1\}$ suffice for DT to predict Y
- Explanation for why $\kappa(0,1,1,1) = \text{N}$?
  - Given $\mathbf{x} = (x_1, x_2, x_3, x_4)$, IF $(x_1 = 0) \wedge (x_2 = 1) \wedge (x_3 = 1)$ THEN $\kappa(\mathbf{x}) = \text{N}$
  - I.e. $\{x_1 = 0, x_2 = 1, x_3 = 1\}$ suffices for DT to predict N

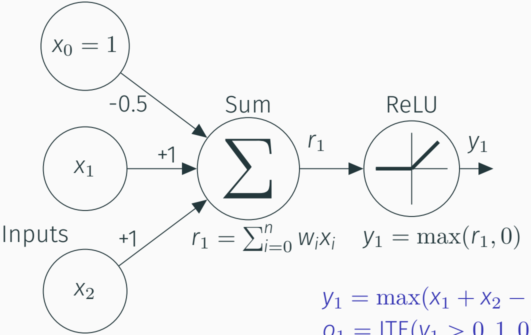
A neural network example



$$y_1 = \max(x_1 + x_2 - 0.5, 0)$$
$$o_1 = \text{ITE}(y_1 > 0, 1, 0)$$

x_1	x_2	r_1	y_1	$\kappa(\mathbf{x})$
0	0	-0.5	0	**0**
0	1	0.5	0.5	**1**
1	0	0.5	0.5	**1**
1	1	1.5	1.5	**1**

A neural network example



x_1	x_2	r_1	y_1	$\kappa(\mathbf{x})$
0	0	-0.5	0	**0**
0	1	0.5	0.5	**1**
1	0	0.5	0.5	**1**
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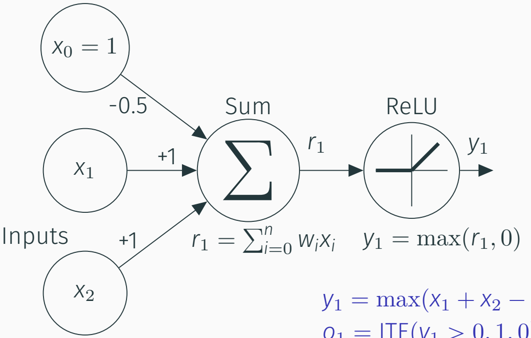
$$y_1 = \max(x_1 + x_2 - 0.5, 0)$$
$$o_1 = \mathsf{ITE}(y_1 > 0, 1, 0)$$

- Explanation for why $\kappa(1,1) = 1$?

A neural network example



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x_1	x_2	r_1	y_1	$\kappa(\mathbf{x})$
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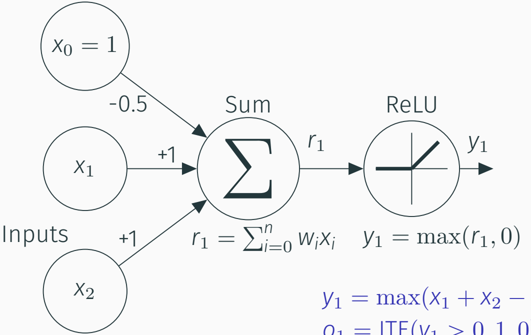
0	1	0.5	0.5	**1**
1	0	0.5	0.5	**1**
1	1	1.5	1.5	**1**

- Explanation for why $\kappa(1, 1) = 1$?
  - Given $\mathbf{x} = (x_1, x_2)$, **IF** $(x_1 = 1)$ **THEN** $\kappa(\mathbf{x}) = 1$
  - I.e. $\{x_1 = 1\}$ suffices for NN to predict **1**

A neural network example



x_1	x_2	r_1	y_1	$\kappa(\mathbf{x})$
0	0	-0.5	0	**0**
0	1	0.5	0.5	**1**
1	0	0.5	0.5	**1**
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$y_1 = \max(x_1 + x_2 - 0.5, 0)$

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- Explanation for why $\kappa(1, 1) = 1$?
  - Given $\mathbf{x} = (x_1, x_2)$, **IF** $(x_1 = 1)$ **THEN** $\kappa(\mathbf{x}) = 1$
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  - Given $\mathbf{x} = (x_1, x_2)$, **IF** $(x_2 = 1)$ **THEN** $\kappa(\mathbf{x}) = 1$
  - I.e. $\{x_2 = 1\}$ suffices for NN to predict Y

An arbitrary classifier

- Classification function:

$$\kappa(x_1, x_2, x_3, x_4) = \neg x_1 \wedge \neg x_2 \vee x_1 \wedge x_2 \wedge x_4 \vee \neg x_1 \wedge x_2 \wedge \neg x_3 \vee \neg x_2 \wedge x_3 \wedge x_4$$

x_1	x_2	x_3	x_4	$\kappa(\mathbf{x})$
0	0	0	0	1
0	0	0	1	1
0	0	1	0	1
0	0	1	1	1
0	1	0	0	1
0	1	0	1	1
0	1	1	0	0
0	1	1	1	0
1	0	0	0	0
1	0	0	1	0
1	0	1	0	0
1	0	1	1	1
1	1	0	0	0
1	1	0	1	1
1	1	1	0	0
1	1	1	1	1

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- Instance: $((0, 0, 0, 0), 1)$

x_1	x_2	x_3	x_4	$\kappa(\mathbf{x})$
0	0	0	0	1
0	0	0	1	1
0	0	1	0	1
0	0	1	1	1
0	1	0	0	1
0	1	0	1	1
0	1	1	0	0
0	1	1	1	0
1	0	0	0	0
1	0	0	1	0
1	0	1	0	0
1	0	1	1	1
1	1	0	0	0
1	1	0	1	1
1	1	1	0	0
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- Instance: $((0, 0, 0, 0), 1)$
- Given $\mathbf{x} = (x_1, x_2, x_3, x_4)$,
  IF $(x_1 = 0) \wedge (x_3 = 0)$ THEN $\kappa(\mathbf{x}) = 1$
- I.e. $\{x_1 = 0, x_3 = 0\}$ suffices for DT to predict 1

x_1	x_2	x_3	x_4	$\kappa(\mathbf{x})$
0	0	0	0	1
0	0	0	1	1
0	0	1	0	1
0	0	1	1	1
0	1	0	0	1
0	1	0	1	1
0	1	1	0	0
0	1	1	1	0
1	0	0	0	0
1	0	0	1	0
1	0	1	0	0
1	0	1	1	1
1	1	0	0	0
1	1	0	1	1
1	1	1	0	0
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Outline – Unit #01

ML Models: Classification & Regression Problems

Basics of (non-symbolic) XAI

Motivation for Explanations

Brief Glimpse of Logic

Reasoning About ML Models

Understanding Intrinsic Interpretability

Standard tools of the trade

- SAT: decision problem for propositional logic
  - Formulas most often represented in CNF
  - There are optimization variants: MaxSAT, PBO, MinSAT, etc.
  - There are quantified variants: QBF, QMaxSAT, etc.

- SMT: decision problem for (decidable) fragments of first-order logic (FOL)
  - There are optimization variants: MaxSMT, etc.
  - There are quantified variants

- MILP: decision/optimization problems defined on conjunctions of linear inequalities over integer & real-valued variables

- CP: constraint programming
  - There are optimization/quantified variants

Standard tools of the trade

- SAT: decision problem for propositional logic
  - Formulas most often represented in CNF
  - There are optimization variants: MaxSAT, PBO, MinSAT, etc.
  - There are quantified variants: QBF, QMaxSAT, etc.

Basic knowledge on SAT & SMT assumed. See links below.

- SMT: decision problem for (decidable) fragments of first-order logic (FOL)
  - There are optimization variants: MaxSMT, etc.
  - There are quantified variants
- MILP: decision/optimization problems defined on conjunctions of linear inequalities over integer & real-valued variables
- CP: constraint programming
  - There are optimization/quantified variants
- Background on SAT/SMT: [BHvMW09]
  - `https://alexeyignatiev.github.io/ssa-school-2019/`
  - `https://alexeyignatiev.github.io/ijcai19tut/`

SAT/SMT/MILP/CP solvers used as oracles – more detail later

- Deciding satisfiability, entailment
- Computing prime implicants/implicates
- Computing MUSes, MCSes
  - Algorithms: Deletion, QuickXplain, Progression, Dichotomic, etc. [MM20]
- Enumeration of MUSes, MCSes
  - Algorithms: Marco, Camus, etc. [LS08, LPMM16]
- Solving MaxSAT, MaxSMT
  - Algorithms: Core-guided, Minimum hitting sets, branch&bound, etc. [MHL+13]
- Solving quantification problems, e.g. QBF
  - Algorithms: Abstraction refinement [JKMC16]

Basic definitions in propositional logic

- Atoms ($\{x, x_1, \ldots\}$) & literals ($x_1, \neg x_1$)
- Well-formed formulas using $\neg$, $\wedge, \vee, \ldots$
- Clause: disjunction of literals
- Term: conjunction of literals
- Conjunctive normal form (CNF): conjunction of clauses
- Disjunctive normal form (DNF): disjunction of terms
- Simple to generalize to more expressive domains

Basic definitions in propositional logic

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- Well-formed formulas using $\neg$, $\wedge, \vee, \ldots$
- Clause: disjunction of literals
- Term: conjunction of literals
- Conjunctive normal form (CNF): conjunction of clauses
- Disjunctive normal form (DNF): disjunction of terms
- Simple to generalize to more expressive domains
- $\mathsf{CO}(\psi(\mathbf{x}))$ decides whether $\psi(\mathbf{x})$ is satisfiable (i.e. whether it is consistent), using an oracle for SAT/SMT/MILP/CP/etc.

Entailment

- Let φ represent some formula, defined on feature space $\mathbb{F}$, and representing a function $\varphi : \mathbb{F} \to \{0, 1\}$
- Let τ represent some other formula, also defined on $\mathbb{F}$, and with $\tau : \mathbb{F} \to \{0, 1\}$

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  - We say that τ **entails** φ, written as $\tau \vDash \varphi$, if:

$$\forall(\mathbf{x} \in \mathbb{F}).[\tau(\mathbf{x}) \to \varphi(\mathbf{x})]$$

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  - We say that $\tau(\mathbf{x})$ is **sufficient** for $\varphi(\mathbf{x})$
- To decide entailment:
  - $\tau \vDash \varphi$ if $\tau(\mathbf{x}) \land \neg\varphi(\mathbf{x})$ is **not** consistent, i.e. $\mathsf{CO}(\tau(\mathbf{x}) \land \neg\varphi(\mathbf{x}))$ does not hold

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  - $\tau \vDash \varphi$ if $\tau(\mathbf{x}) \land \neg\varphi(\mathbf{x})$ is **not** consistent, i.e. **CO**$(\tau(\mathbf{x}) \land \neg\varphi(\mathbf{x}))$ does not hold
- An example:
  - $\mathbb{F} = \{0, 1\}^2$
  - $\varphi(x_1, x_2) = x_1 \lor \neg x_2$
  - Clearly, $x_1 \vDash \varphi$ and $\neg x_2 \vDash \varphi$
  - Also, **CO**$(x_1 \land (\neg x_1 \land x_2))$ does not hold

Entailment

- Let φ represent some formula, defined on feature space $\mathbb{F}$, and representing a function $\varphi : \mathbb{F} \to \{0,1\}$
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- An example:
  - $\mathbb{F} = \{0,1\}^2$
  - $\varphi(x_1, x_2) = x_1 \lor \neg x_2$
  - Clearly, $x_1 \vDash \varphi$ and $\neg x_2 \vDash \varphi$
  - Also, $\mathsf{CO}(x_1 \land (\neg x_1 \land x_2))$ does not hold
- Another example:
  - $\mathbb{F} = \{0,1\}^3$
  - $\varphi(x_1, x_2, x_3) = x_1 \land x_2 \lor x_1 \land x_3$
  - Clearly, $x_1 \land x_2 \vDash \varphi$ and $x_1 \land x_3 \vDash \varphi$
  - Also, $\mathsf{CO}(x_1 \land x_2 \land ((\neg x_1 \lor \neg x_2) \land (\neg x_1 \lor \neg x_3)))$ does not hold

Entailment & explanations – how do we construct explanations?

- Classification function:

$$\kappa(x_1, x_2, x_3, x_4) = \neg x_1 \wedge \neg x_2 \vee x_1 \wedge x_2 \wedge x_4 \vee \neg x_1 \wedge x_2 \wedge \neg x_3 \vee \neg x_2 \wedge x_3 \wedge x_4$$

  - Instance: $((0, 1, 0, 0), 1)$

x_1	x_2	x_3	x_4	$\kappa(\mathbf{x})$
0	0	0	0	1
0	0	0	1	1
0	0	1	0	1
0	0	1	1	1
0	1	0	0	1
0	1	0	1	1
0	1	1	0	0
0	1	1	1	0
1	0	0	0	0
1	0	0	1	0
1	0	1	0	0
1	0	1	1	1
1	1	0	0	0
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  - Instance: $((0, 1, 0, 0), 1)$

- Localized explanation: any irreducible conjunction of literals, consistent with $\mathbf{v}$, and that entails the prediction

x_1	x_2	x_3	x_4	$\kappa(\mathbf{x})$
0	0	0	0	1
0	0	0	1	1
0	0	1	0	1
0	0	1	1	1
0	1	0	0	1
0	1	0	1	1
0	1	1	0	0
0	1	1	1	0
1	0	0	0	0
1	0	0	1	0
1	0	1	0	0
1	0	1	1	1
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1	1	0	1	1
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Entailment & explanations – how do we construct explanations?

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- Localized explanation: any irreducible conjunction of literals, consistent with $\mathbf{v}$, and that entails the prediction
  - Given $\mathbf{x} = (x_1, x_2, x_3, x_4)$,
    IF $(x_1 = 0) \wedge (x_3 = 0)$ THEN $\kappa(\mathbf{x}) = 1$

x_1	x_2	x_3	x_4	$\kappa(\mathbf{x})$
0	0	0	0	1
0	0	0	1	1
0	0	1	0	1
0	0	1	1	1
0	1	0	0	1
0	1	0	1	1
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0	1	1	1	0
1	0	0	0	0
1	0	0	1	0
1	0	1	0	0
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  - Instance: $((0, 1, 0, 0), 1)$

- Localized explanation: any irreducible conjunction of literals, consistent with $\mathbf{v}$, and that entails the prediction
  - Given $\mathbf{x} = (x_1, x_2, x_3, x_4)$,
    IF $(x_1 = 0) \wedge (x_3 = 0)$ THEN $\kappa(\mathbf{x}) = 1$

- Global explanation: any irreducible conjunction of literals, that is consistent, and that entails the prediction

x_1	x_2	x_3	x_4	$\kappa(\mathbf{x})$
0	0	0	0	1
0	0	0	1	1
0	0	1	0	1
0	0	1	1	1
0	1	0	0	1
0	1	0	1	1
0	1	1	0	0
0	1	1	1	0
1	0	0	0	0
1	0	0	1	0
1	0	1	0	0
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1	1	0	0	0
1	1	0	1	1
1	1	1	0	0
1	1	1	1	1

- Classification function:

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  - Instance: $((0,1,0,0),1)$

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  - Given $\mathbf{x} = (x_1, x_2, x_3, x_4)$,
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x_1	x_2	x_3	x_4	$\kappa(\mathbf{x})$
0	0	0	0	1
0	0	0	1	1
0	0	1	0	1
0	0	1	1	1
0	1	0	0	1
0	1	0	1	1
0	1	1	0	0
0	1	1	1	0
1	0	0	0	0
1	0	0	1	0
1	0	1	0	0
1	0	1	1	1
1	1	0	0	0
1	1	0	1	1
1	1	1	0	0
1	1	1	1	1

Outline – Unit #01

Decision sets with boolean features

- Example ML model:

  Features: $x_1, x_2, x_3, x_4 \in \{0, 1\}$ (boolean)

  Rules:

  $$\begin{array}{llll} \text{IF} & x_1 \land \neg x_2 \land x_3 & \text{THEN} & \text{predict } \boxplus \\ \text{IF} & x_1 \land \neg x_3 \land x_4 & \text{THEN} & \text{predict } \boxminus \\ \text{IF} & x_3 \land x_4 & \text{THEN} & \text{predict } \boxminus \end{array}$$

Decision sets with boolean features

- Example ML model:

  Features: $x_1, x_2, x_3, x_4 \in \{0, 1\}$ (boolean)

  Rules:

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  - A formalization:

$$\begin{array}{l} y_{p,1} \leftrightarrow (x_1 \land \neg x_2 \land x_3) \land \\ y_{n,1} \leftrightarrow (x_1 \land \neg x_3 \land x_4) \land \\ y_{n,2} \leftrightarrow (x_3 \land x_4) \land (y_p \leftrightarrow y_{p,1}) \land \\ (y_n \leftrightarrow (y_{n,1} \lor y_{n,2})) \land (y_p) \land (y_n) \end{array}$$

  ... and solve with SAT solver (after clausification) [Tse68, PG86]

  Or use PySAT [IMM18]

  ∴ There exists a model iff there exists a point in feature space yielding both predictions

Decision sets with ordinal features

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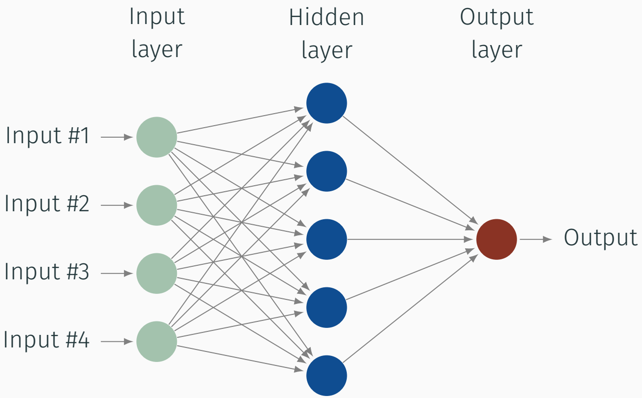
$$y_p \leftrightarrow (2x_1 + x_2 > 0) \;\wedge\; y_n \leftrightarrow (2x_1 - x_2 \leqslant 0) \;\wedge\; (y_p) \;\wedge\; (y_n)$$

... and solve with SMT solver (many alternatives)

∴ There exists a model iff there exists a point in feature space yielding both predictions

Neural networks



- Each layer (except first) viewed as a **block**, and
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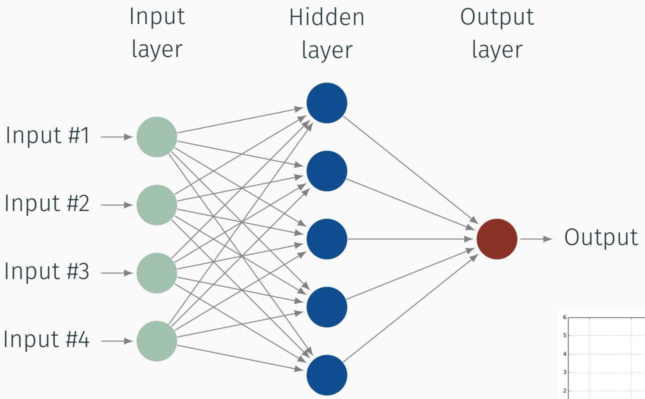

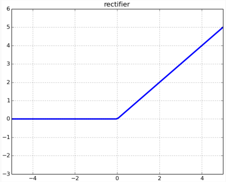

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- Each unit uses a **ReLU** activation function

[NH10]

Encoding NNs using MILP

Computation for a NN ReLU **block**, in two steps:

$$\mathbf{x}' = \mathbf{A} \cdot \mathbf{x} + \mathbf{b}$$

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Encoding each **block**: [FJ18]

$$\sum_{j=1}^{n} a_{i,j} x_j + b_i = y_i - s_i$$
$$z_i = 1 \rightarrow y_i \leqslant 0$$
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$$y_i \geqslant 0, s_i \geqslant 0, z_i \in \{0, 1\}$$

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Modeling ML models with logic is not only possible but also simple !

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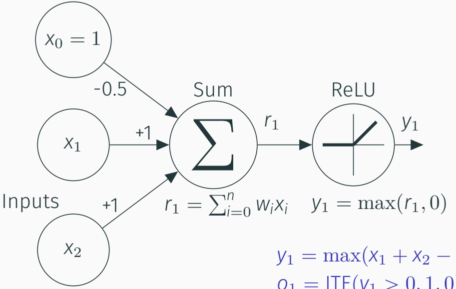

x_1	x_2	r_1	y_1	o_1
0	0	-0.5	0	0
1	0	0.5	0.5	1
0	1	0.5	0.5	1
1	1	1.5	1.5	1

$y_1 = \max(x_1 + x_2 - 0.5, 0)$
$o_1 = \text{ITE}(y_1 > 0, 1, 0)$

MILP encoding:

$x_1 + x_2 - 0.5 = y_1 - s_1$
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$x_1, x_2, z_1, o_1 \in \{0, 1\}$
$y_1, s_1 \geqslant 0$

Instance: $(\mathbf{x}, c) = ((1, 0), 1)$

$1 + 0 - 0.5 = 0.5 - 0$
$1 \vee 0.5 \leqslant 0$
$0 \vee 0 \leqslant 0$
$1 = (0.5 > 0)$
$x_1 = 1, x_2 = 0, z_1 = 0, o_1 = 1$
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Checking: $\mathbf{x} = (0, 0)$

$0 + 0 - 0.5 = 0 - 0.5$
$0 \vee 0 \leqslant 0$
$1 \vee 0.5 \leqslant 0$
$0 = (0 > 0)$
$x_1 = 0, x_2 = 0, z_1 = 1, o_1 = 0$
$y_1 = 0, s_1 = 0.5$

Outline – Unit #01

What is intrinsic interpretability?

- Goal is to deploy *interpretable* ML models [Rud19, Mol20, RCC+22, Rud22]
  - E.g. Decision trees, decision lists, decision sets, etc.
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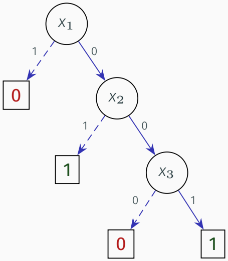

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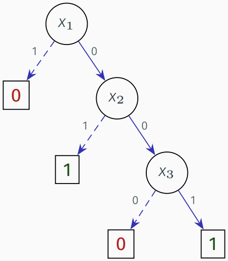

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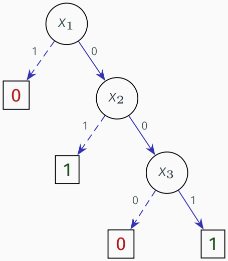

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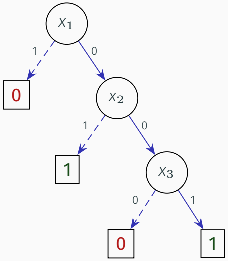

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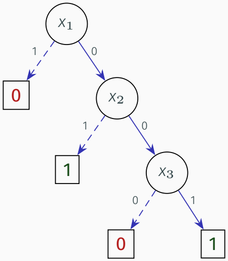

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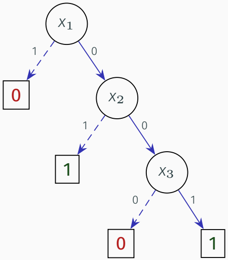

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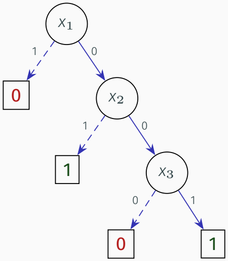

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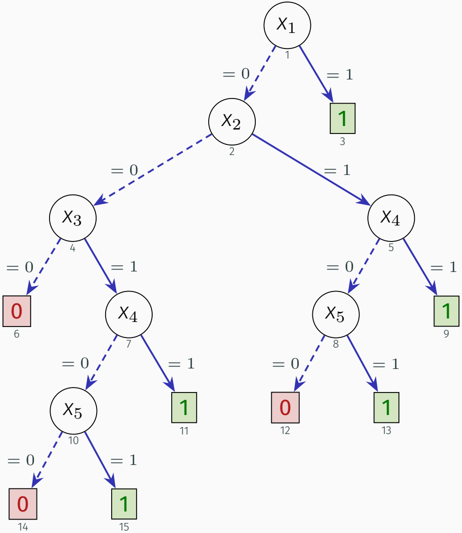
  - $\{1,3\}$ is easier to grasp; also, it is irreducible

Are *interpretable* models really interpretable? – DTs



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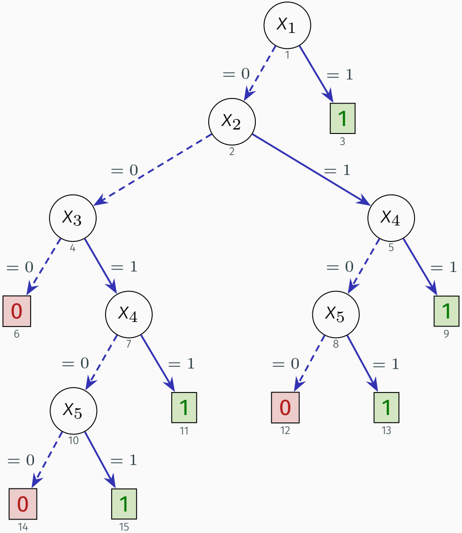

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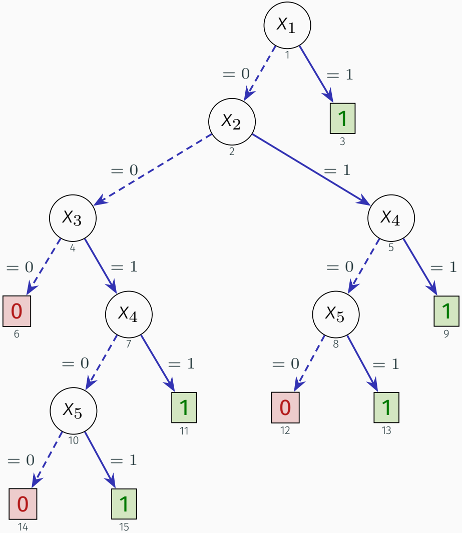

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x_3	x_5	x_1	x_2	x_4	$\kappa(\mathbf{x})$
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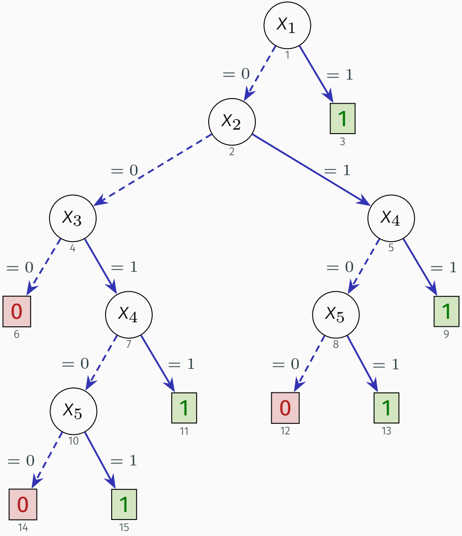

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$\therefore$ fixing $\{3, 5\}$ suffices for the prediction

Compare with $\{1, 2, 3, 4, 5\}$...

R_1 :	IF	$(x_1 \land x_3)$	THEN	$\kappa(\mathbf{x}) = 1$
R_2 :	ELSE IF	$(x_2 \land x_4 \land x_6)$	THEN	$\kappa(\mathbf{x}) = 0$
R_3 :	ELSE IF	$(\neg x_1 \land x_3)$	THEN	$\kappa(\mathbf{x}) = 1$
R_4 :	ELSE IF	$(x_4 \land x_6)$	THEN	$\kappa(\mathbf{x}) = 0$
R_5 :	ELSE IF	$(\neg x_1 \land \neg x_3)$	THEN	$\kappa(\mathbf{x}) = 1$
R_6 :	ELSE IF	(x_6)	THEN	$\kappa(\mathbf{x}) = 0$
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Are *interpretable* models really interpretable? – DLs

[MSI23]

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  - Some questions:
    - Would average human decision maker be able to understand the irreducible set $\{3,4,6\}$?

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- Fixing $\{3,4,6\}$ suffices for the prediction
  - Why?
    - We need 3 (or 1) so that R_1 cannot fire
    - With 3, we do not need 2, since with 4 and 6 fixed, then R_4 is guaranteed to fire
  - Some questions:
    - Would average human decision maker be able to understand the irreducible set $\{3,4,6\}$?
    - Would he/she be able to compute the set $\{3,4,6\}$, by manual inspection?

Questions?

Lecture 02

Recapitulate first lecture

- ML models: classification & regression

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- Glimpse of heuristic XAI

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- Apparent difficulties with explaining interpretable models

Plan for this course

- Lecture 01 – units:
  - #01: Foundations
- Lecture 02 – units:
  - #02: Principles of symbolic XAI – feature selection
  - #03: Tractability in symbolic XAI (& myth of interpretability)
- Lecture 03 – units:
  - #04: Intractability in symbolic XAI (& myth of model-agnostic XAI)
  - #05: Explainability queries
- Lecture 04 – units:
  - #06: Advanced topics
- Lecture 05 – units:
  - #07: Principles of symbolic XAI – feature attribution (& myth of Shapley values in XAI)
  - #08: Conclusions & research directions

Unit #02

Principles of Symbolic XAI – Feature Selection

Outline – Unit #02

Definitions of Explanations

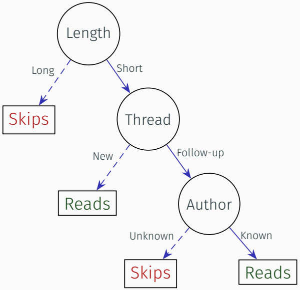
Duality Properties

Computational Problems

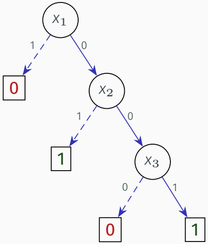
What is an explanation?

- Notation:

Original DT [PM17]



Rewritten DT



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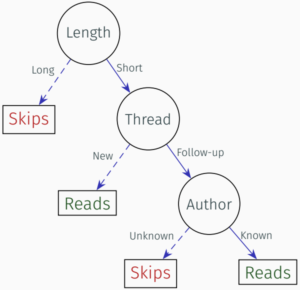

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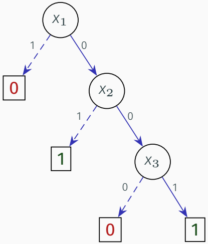

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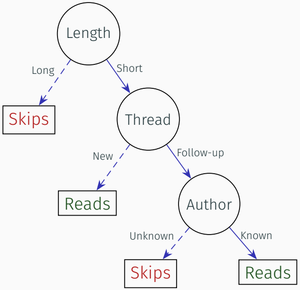

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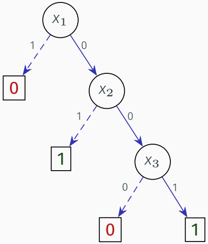

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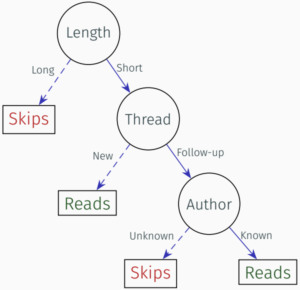

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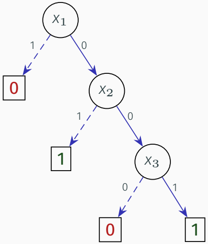

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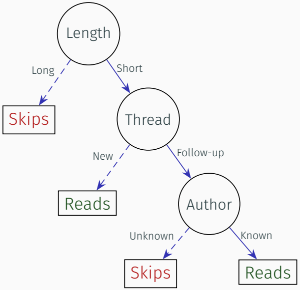

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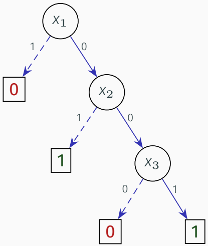

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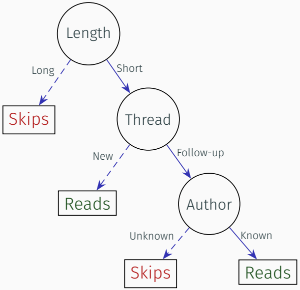

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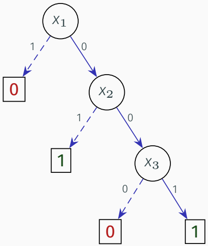

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    - One possible explanation is $\{\neg x_1, \neg x_2, x_3\}$ or simply $\{1, 2, 3\}$

The similarity predicate

[Mar24]

- Recall ML models for classification & regression:
  - Classification: $\mathcal{M}_C = (\mathcal{F}, \mathbb{F}, \mathcal{K}, \kappa)$
  - Regression: $\mathcal{M}_R = (\mathcal{F}, \mathbb{F}, \mathbb{V}, \rho)$
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- Bottom line:
  Reason about symbolic explainability by abstracting away type of ML model

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- Finding one AXp (example algorithm; many more exist): [MM20]
  - Let $\mathcal{X} = \mathcal{F}$, i.e. fix all features
  - Invariant: $\mathsf{WAXp}(\mathcal{X})$ must hold. **Why?**
  - Analyze features in any order, one feature i at a time
    - If $\mathsf{WAXp}(\mathcal{X} \setminus \{i\})$ holds, then remove i from $\mathcal{X}$, i.e. i becomes free

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Recap weak AXp: $\forall(\mathbf{x} \in \mathbb{F}). \bigwedge_{j \in \mathcal{X}} (x_j = v_j) \rightarrow (\sigma(\mathbf{x}))$

A simple example – AXp's

- Classifier:

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- Point $\mathbf{v} = (0, 0, 0, 1)$ with prediction $\kappa(\mathbf{v}) = 1$. AXp?
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  - **Obs:** for some classes of classifiers, poly-time algorithms exist

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More notation

- Notation $\mathbf{x}_{\mathcal{S}} = \mathbf{v}_{\mathcal{S}}$:

$$[\mathbf{x}_{\mathcal{S}} = \mathbf{v}_{\mathcal{S}}] \quad \equiv \quad \bigwedge_{i \in \mathcal{S}} (x_i = v_i)$$

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- Using probabilities, non-real-valued features: [WMHK21, IHI+22, ABOS22, IHI+23]

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- Finding one CXp: [MM20]
  - Let $\mathcal{Y} = \mathcal{F}$, i.e. free all features
  - Invariant: $\mathsf{WCXp}(\mathcal{Y})$ must hold. **Why?**
  - Analyze features in any order, one feature i at a time
    - If $\mathsf{WCXp}(\mathcal{Y}\backslash\{i\})$ holds, then remove i from $\mathcal{Y}$, i.e. i is becomes fixed

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- **Obs:** AXp is MHS of CXp and vice-versa...

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- Definition of CXp remains unchanged

Detour: global explanations

[INM19b]

- AXps and CXps are defined locally (because of $\mathbf{v}$) but hold globally
  - Localized explanations
  - Can be viewed as attempt at formalizing local explanations [RSG16, LL17, RSG18]
- One can define explanations without picking a given point in feature space
  - Let $q \in \mathbb{T}$, and refefine the similarity predicate:
    - Classification: $\sigma(\mathbf{x}) = [\kappa(\mathbf{x}) = q]$
    - Regression: $\sigma(\mathbf{x}) = [|\kappa(\mathbf{x}) - q| \leqslant \delta]$, δ is user-specified
  - Let $\mathbb{L} = \{(x_i = v_i) \,|\, i \in \mathcal{F} \wedge v_i \in \mathbb{V}\}$
  - Let $\mathcal{S} \subsetneq \mathbb{L}$ be a subset of literals that does not repeat features, i.e. $\mathcal{S}$ is not inconsistent
  - Then, $\mathcal{S}$ is a global AXp if,

$$\forall(\mathbf{x} \in \mathbb{F}). \bigwedge\nolimits_{(x_i=v_i)\in\mathcal{S}}(x_i = v_i) \rightarrow (\sigma(\mathbf{x}))$$

  - Counterexamples are minimal hitting sets of global AXps and vice-versa [INM19b]

Duality in explainability – basic results

[INAM20, Mar22]

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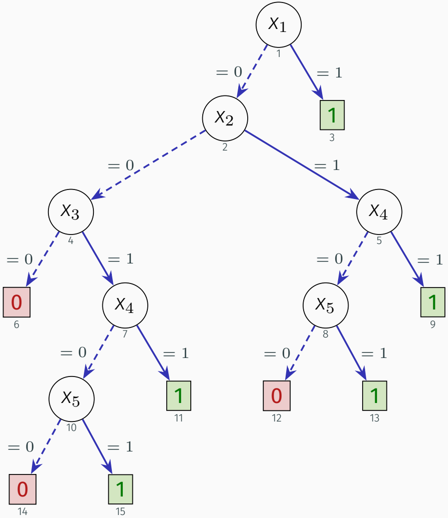

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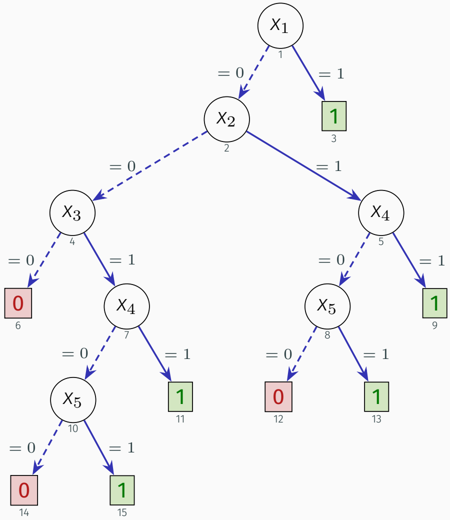

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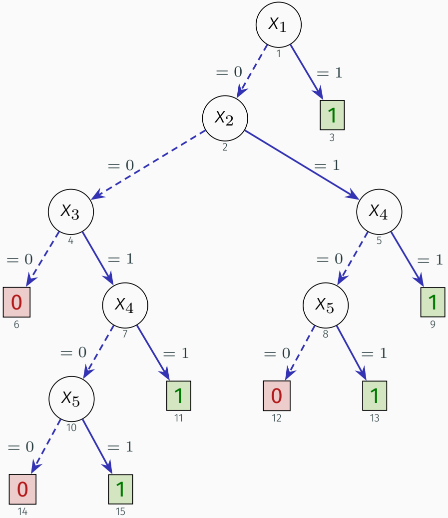

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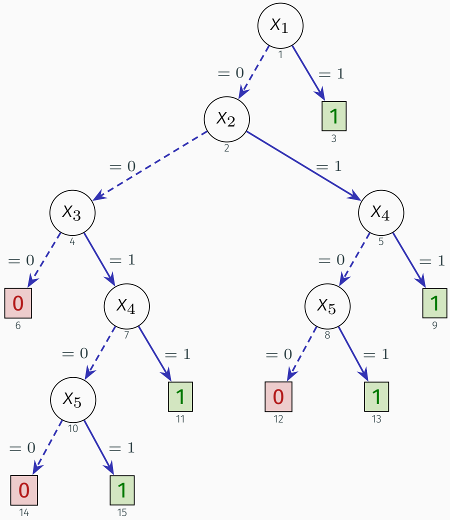

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[INAM20, Mar22]

- **Claim:**
  $\mathcal{S} \subseteq \mathcal{F}$ is an AXp iff it is a minimal hitting set (MHS) of the set of CXps
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- An example, $(\mathbf{v}, c) = ((0, 0, 1, 0, 1), 1)$:
  - AXps: $\{\{3, 5\}\}$
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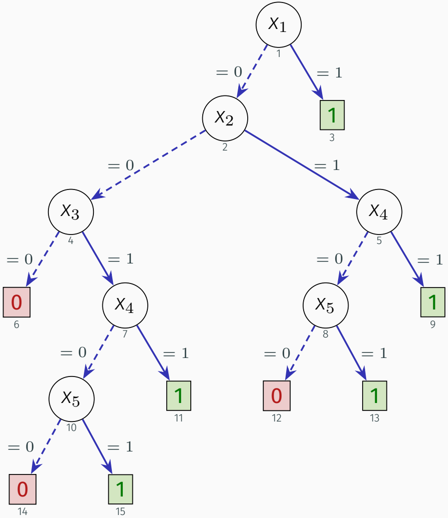

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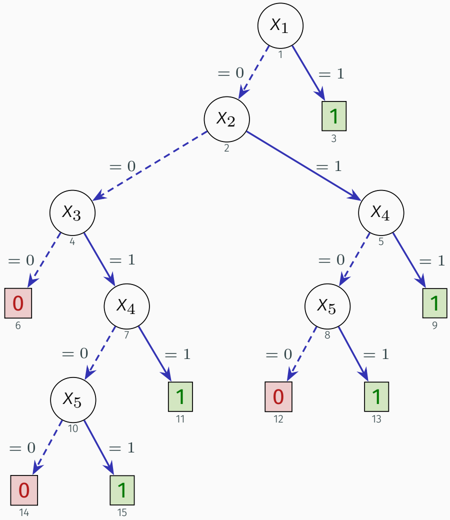

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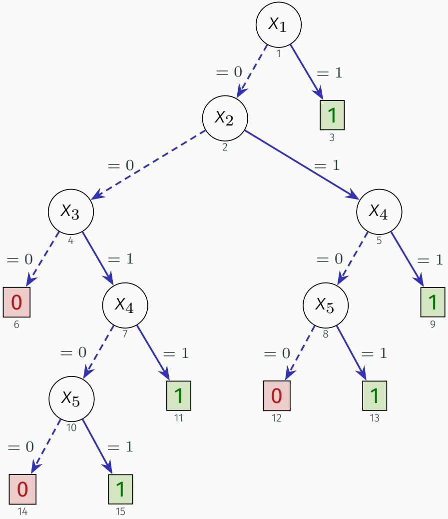
  - BTW,
    - $\{2, 5\}$ is not a CXp
    - $\{1, 2, 3, 4, 5\}$, $\{1, 2, 3, 5\}$ and $\{1, 3, 5\}$ are not AXps



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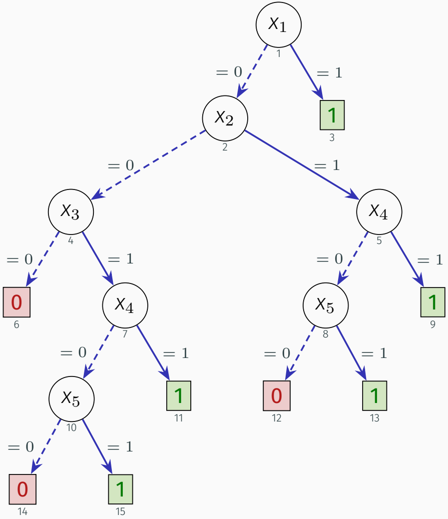
    - Why?



Outline – Unit #02

Definitions of Explanations

Duality Properties

Computational Problems

Computational problems in (formal) explainability

- Compute one abductive/contrastive explanation

Computational problems in (formal) explainability

- Compute one abductive/contrastive explanation
- Enumerate all abductive/contrastive explanations

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- Monotone predicates for WAXp & WCXp:

$$\mathbb{P}_{\text{axp}}(\mathcal{S}) \triangleq \neg\,\mathsf{CO}\left(\left[\!\left[\left(\bigwedge\nolimits_{i\in\mathcal{S}}(x_i = v_i)\right) \land (\neg\sigma(\mathbf{x}))\right]\!\right]\right) \qquad \mathbb{P}_{\text{cxp}}(\mathcal{S}) \triangleq \mathsf{CO}\left(\left[\!\left[\left(\bigwedge\nolimits_{i\in\mathcal{F}\setminus\mathcal{S}}(x_i = v_i)\right) \land (\neg\sigma(\mathbf{x}))\right]\!\right]\right)$$

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Input: Predicate $\mathbb{P}$, parameterized by $\mathcal{T}$, $\mathcal{M}$

Output: One XP $\mathcal{S}$

1: **procedure** oneXP($\mathbb{P}$)

2: $\quad\mathcal{S} \leftarrow \mathcal{F}$ ▷ Initialization: $\mathbb{P}(\mathcal{S})$ holds

3: $\quad$**for** $i \in \mathcal{F}$ **do** ▷ Loop invariant: $\mathbb{P}(\mathcal{S})$ holds

4: $\quad\quad$**if** $\mathbb{P}(\mathcal{S}\setminus\{i\})$ **then**

5: $\quad\quad\quad\mathcal{S} \leftarrow \mathcal{S}\setminus\{i\}$ ▷ Update $\mathcal{S}$ only if $\mathbb{P}(\mathcal{S}\setminus\{i\})$ holds

6: $\quad$**return** $\mathcal{S}$ ▷ Returned set $\mathcal{S}$: $\mathbb{P}(\mathcal{S})$ holds

Computing one AXp/CXp

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```
procedure oneXP(ℙ)
    𝒮 ← ℱ                           ▷ Initialization: ℙ(𝒮) holds
    for i ∈ ℱ do                    ▷ Loop invariant: ℙ(𝒮) holds
        if ℙ(𝒮\{i}) then
            𝒮 ← 𝒮\{i}               ▷ Update 𝒮 only if ℙ(𝒮\{i}) holds
    return 𝒮                        ▷ Returned set 𝒮: ℙ(𝒮) holds
```

Exploiting MSMP, i.e. basic algorithm used for different problems.

Detour: More Connections with Automated Reasoning

Prime implicants & implicates

- A conjunction of literals π (which will be viewed as a set of literals where convenient) is a prime implicant of some function φ if,
  1. $\pi \vDash \varphi$
  2. For any $\pi' \subsetneq \pi$, $\pi' \nvDash \varphi$

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  - Example:
    - $\mathbb{F} = \{0,1\}^3$
    - $\varphi(x_1, x_2, x_3) = x_1 \wedge x_2 \vee x_1 \wedge x_3$
    - Clearly, $x_1 \wedge x_2 \vDash \varphi$
    - Also, $x_1 \nvDash \varphi$ and $x_2 \nvDash \varphi$

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  - Example:
    - $\mathbb{F} = \{0,1\}^3$
    - $\varphi(x_1, x_2, x_3) = x_1 \land x_2 \lor x_1 \land x_3$
    - Clearly, $x_1 \land x_2 \vDash \varphi$
    - Also, $x_1 \nvDash \varphi$ and $x_2 \nvDash \varphi$
- A disjunction of literals η (also viewed as a set of literals where convenient) is a prime implicate of some function φ if
  1. $\varphi \vDash \eta$
  2. For any $\eta' \subsetneq \eta$, $\varphi \nvDash \eta'$

Reasoning about inconsistency

[MM20]

- Formula $\mathcal{T} = \mathcal{B} \cup \mathcal{S}$, with
  - $\mathcal{B}$: background knowledge (base), i.e. hard constraints
  - $\mathcal{S}$: additional (inconsistent) knowledge, i.e. soft constraints
  - And, $\mathcal{T} \vDash \bot$
  - E.g. $\mathcal{B} = \{(x_1 \lor x_2), (x_1 \lor \neg x_3)\}$, $\mathcal{S} = \{(\neg x_1), (\neg x_2), (x_3)\}$

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- Minimal unsatisfiable subset (MUS):
  - Subset-minimal set $\mathcal{U} \subseteq \mathcal{S}$, s.t. $\mathcal{B} \cup \mathcal{U} \vDash \bot$
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  - Subset-minimal set $\mathcal{C} \subseteq \mathcal{S}$, s.t. $\mathcal{B} \cup (\mathcal{S} \setminus \mathcal{C}) \not\models \perp$
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- Duality:
  - MUSes are minimal-hitting sets (MHSes) of the MCSes, and vice-versa [Rei87]

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- Duality:
  - MUSes are minimal-hitting sets (MHSes) of the MCSes, and vice-versa [Rei87]
- Variants:
  - Smallest(-cost) MCS, i.e. complement of maximum(-cost) satisfiability (MaxSAT)
  - Smallest(-cost) MUS

Computing AXps (resp. CXps) as MUSes (resp. MCSes)

- Recap:

$$\begin{aligned}\mathsf{WAXp}(\mathcal{X}) &:= \forall(\mathbf{x} \in \mathbb{F}).\bigwedge\nolimits_{j\in\mathcal{X}}(x_j = v_j) \rightarrow (\sigma(\mathbf{x}))\\ \mathsf{WCXp}(\mathcal{Y}) &:= \exists(\mathbf{x} \in \mathbb{F}).\bigwedge\nolimits_{j\notin\mathcal{Y}}(x_j = v_j) \land (\neg\sigma(\mathbf{x}))\end{aligned}$$

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- Let,
  - Hard constraints, $\mathcal{B}$:

$$\mathcal{B} := \wedge_{i\in\mathcal{F}}\,(s_i \rightarrow (x_i = v_i)) \land \mathsf{Encode}_{\mathcal{T}}(\neg\sigma(\mathbf{x}))$$

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  - Soft constraints: $\mathcal{S}=\{s_i\,|\,i\in\mathcal{F}\}$

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- **Claim**: Each MUS of $(\mathcal{B},\mathcal{S})$ is an AXp & each MCS of $(\mathcal{B},\mathcal{S})$ is a CXp

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- **Claim**: Each MUS of $(\mathcal{B}, \mathcal{S})$ is an AXp & each MCS of $(\mathcal{B}, \mathcal{S})$ is a CXp
  - Can use MUS/MCS algorithms for AXps/CXps

Unit #03

Tractability in Symbolic XAI

Outline – Unit #03

Explanations for Decision Trees

XAI Queries for DTs

Myth #01: Intrinsic Interpretability

Detour: From Decision Trees to Explained Decision Sets

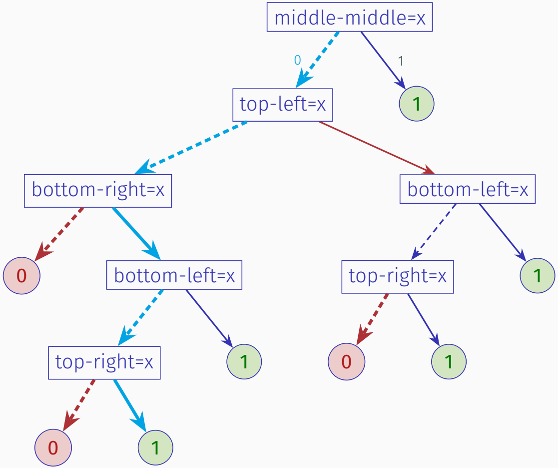
Explanations for Decision Graphs

Explanations for Monotonic Classifiers

Review examples

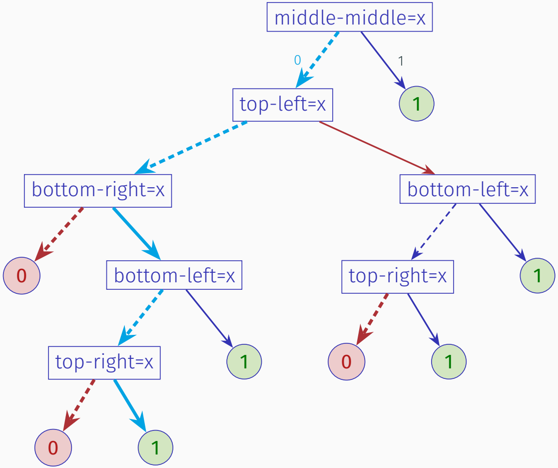
DT explanations

[IIM20]



DT explanations

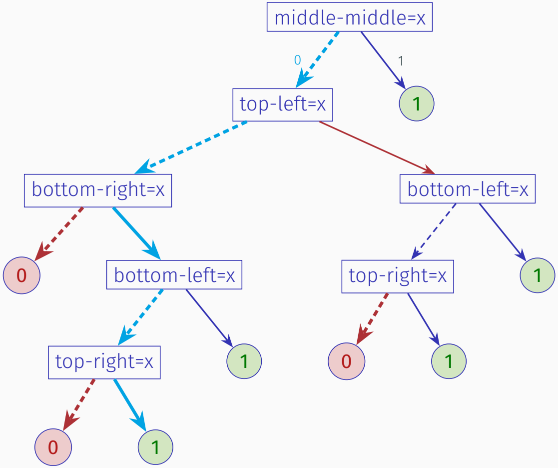
[IIM20]



- Run PI-explanation algorithm based on NP-oracles
  - Worst-case exponential time

DT explanations

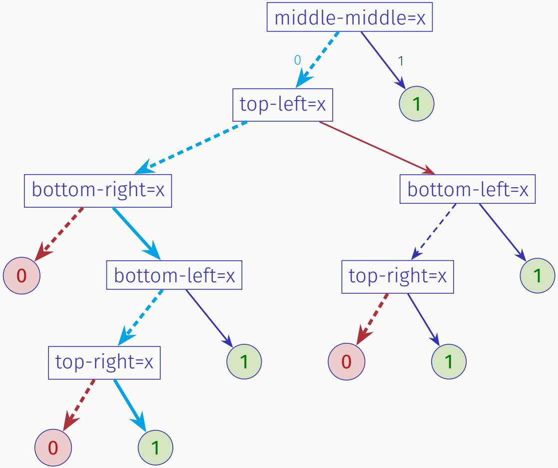
[IIM20]



- Run PI-explanation algorithm based on NP-oracles
  - Worst-case exponential time
- For prediction 1, it suffices to ensure all paths with prediction 0 remain inconsistent

DT explanations in polynomial time

[IIM20]



- Run PI-explanation algorithm based on NP-oracles
  - Worst-case exponential time
- For prediction 1, it suffices to ensure all paths with prediction 0 remain inconsistent
  - I.e. find a subset-minimal hitting set of all 0 paths; these are the features to keep
    - E.g. BR and TR suffice for prediction
  - Well-known to be solvable in polynomial time [EG95]

Outline – Unit #03

Explanations for Decision Trees

XAI Queries for DTs

Myth #01: Intrinsic Interpretability

Detour: From Decision Trees to Explained Decision Sets

Explanations for Decision Graphs

Explanations for Monotonic Classifiers

Review examples

Answering queries in DTs

- Finding one AXp in polynomial-time – covered

Answering queries in DTs

- Finding one AXp in polynomial-time – covered
- Finding one CXp in polynomial-time

Answering queries in DTs

- Finding one AXp in polynomial-time – covered
- Finding one CXp in polynomial-time
- Finding all CXps in polynomial-time

Answering queries in DTs

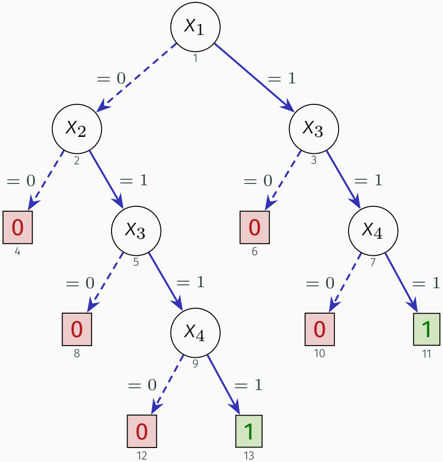
- Finding one AXp in polynomial-time – covered
- Finding one CXp in polynomial-time
- Finding all CXps in polynomial-time; hence, finding one CXp also in polynomial-time

Answering queries in DTs

- Finding one AXp in polynomial-time – covered
- Finding one CXp in polynomial-time
- Finding all CXps in polynomial-time; hence, finding one CXp also in polynomial-time
- Practically efficient enumeration of AXps – later

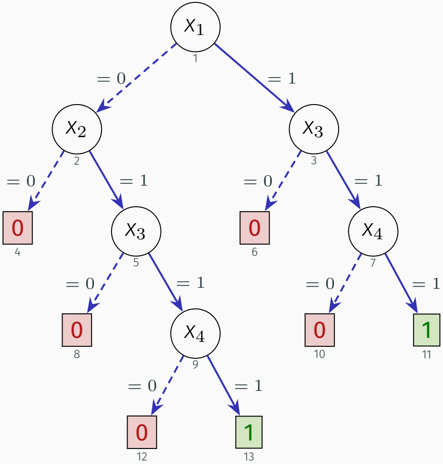
Finding all CXps in polynomial-time

- Basic algorithm:
  - $\mathcal{L} = \emptyset$



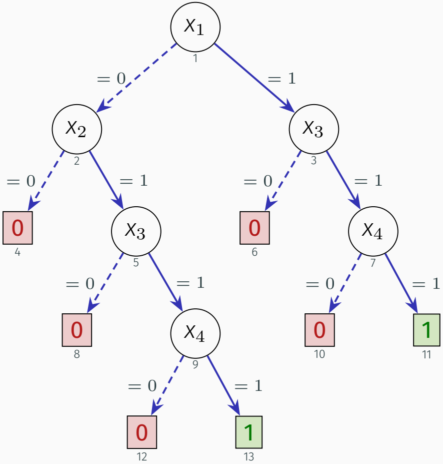
Finding all CXps in polynomial-time

- Basic algorithm:
  - $\mathcal{L} = \emptyset$
  - For each leaf node not predicting q:



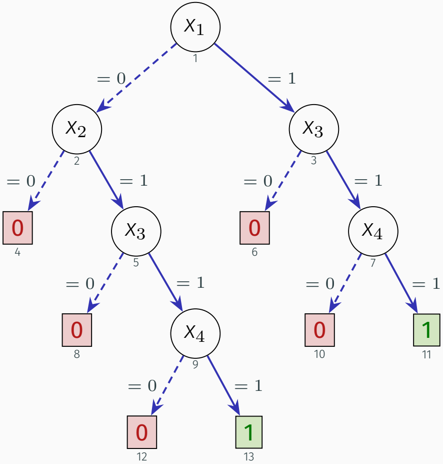
Finding all CXps in polynomial-time

- Basic algorithm:
  - $\mathcal{L} = \emptyset$
  - For each leaf node not predicting q:
    - $\mathcal{I}$: features with literals inconsistent with $\mathbf{v}$



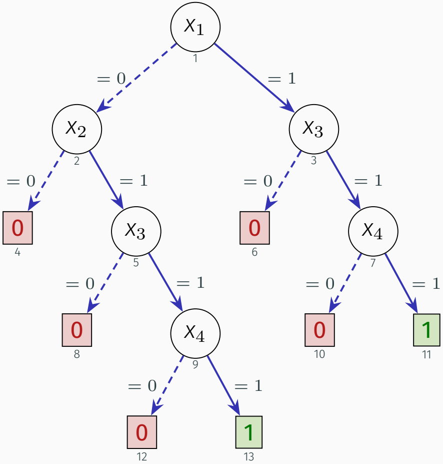
Finding all CXps in polynomial-time

- Basic algorithm:
  - $\mathcal{L} = \emptyset$
  - For each leaf node not predicting q:
    - $\mathcal{I}$: features with literals inconsistent with $\mathbf{v}$
    - Add $\mathcal{I}$ to $\mathcal{L}$



Finding all CXps in polynomial-time

- Basic algorithm:
  - $\mathcal{L} = \emptyset$
  - For each leaf node not predicting q:
    - $\mathcal{I}$: features with literals inconsistent with $\mathbf{v}$
    - Add $\mathcal{I}$ to $\mathcal{L}$
  - Remove from $\mathcal{L}$ non-minimal sets



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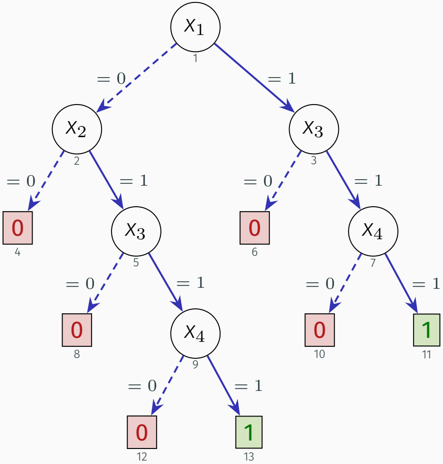

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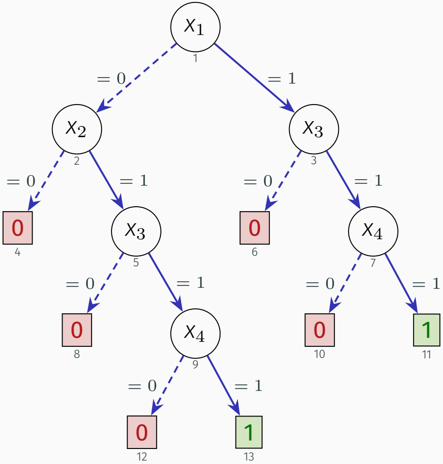

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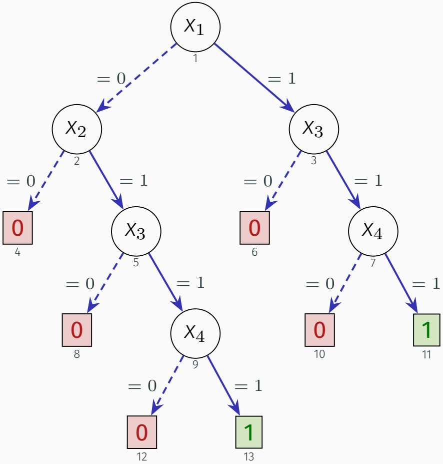

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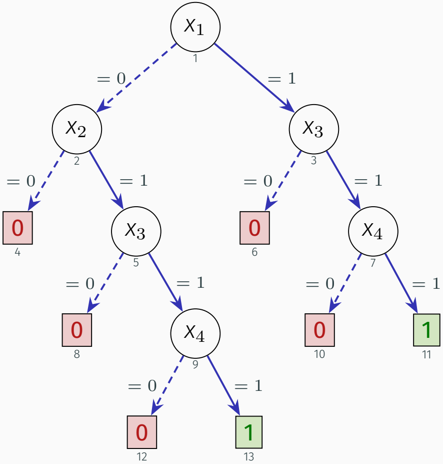

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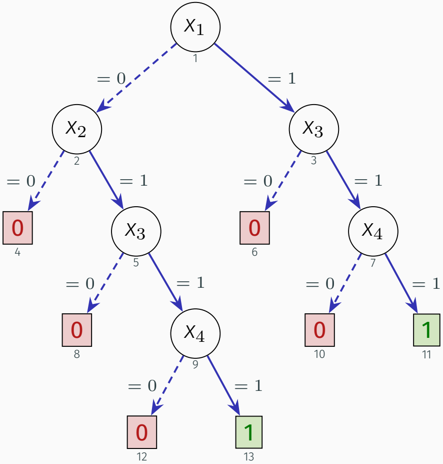

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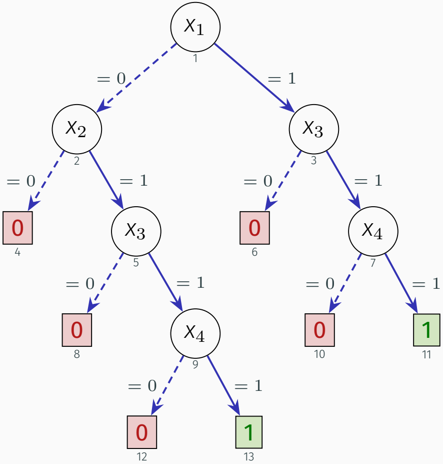

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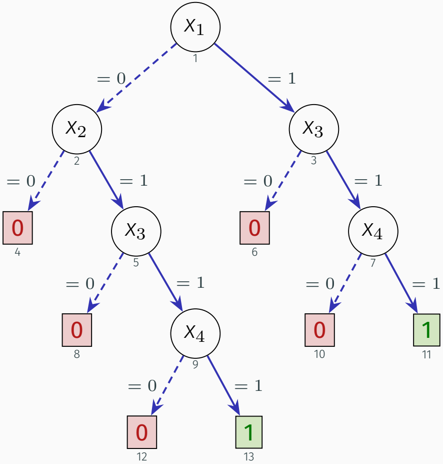

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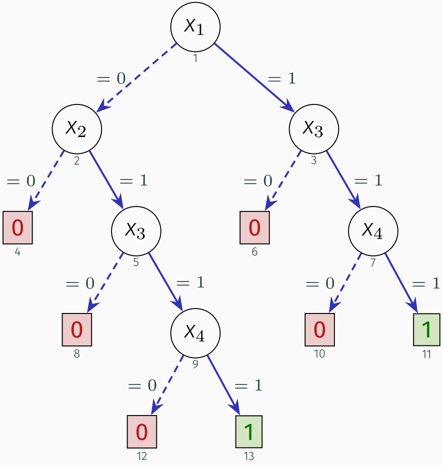

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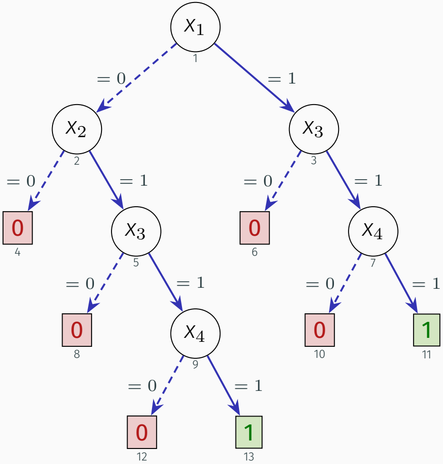

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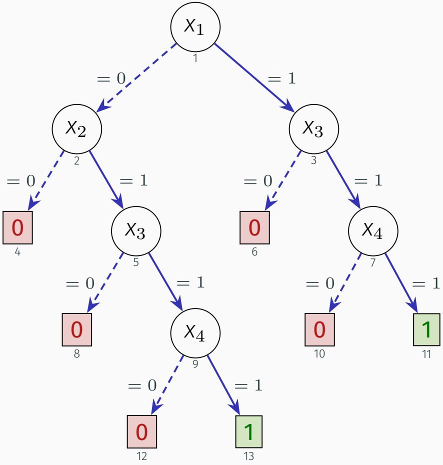
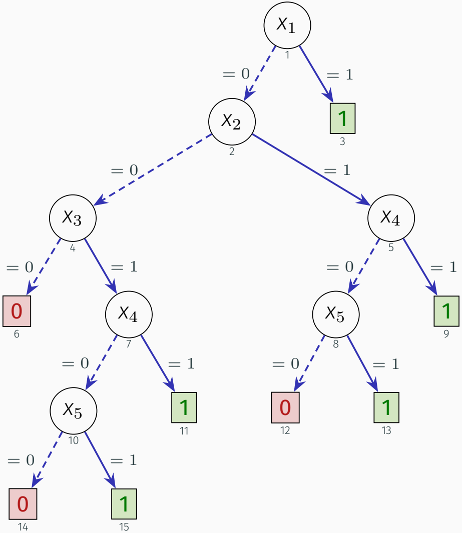
  - AXps: $\{\{1,3,4\},\{2,3,4\}\}$, by computing all MHSes



Outline – Unit #03

Are *interpretable* models really interpretable? – DTs



- Case of **optimal** decision tree (DT) [HRS19]
- Explanation for $(0, 0, 1, 0, 1)$, with prediction 1?

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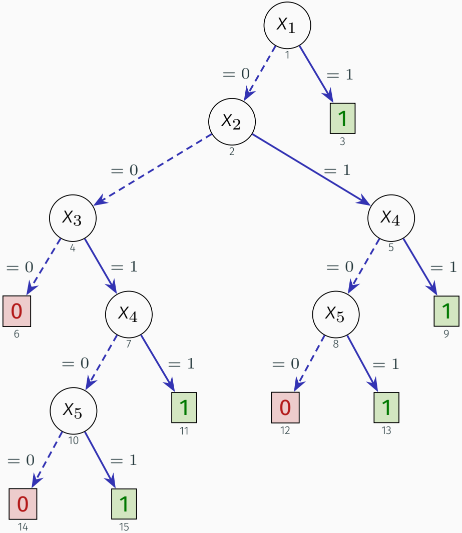

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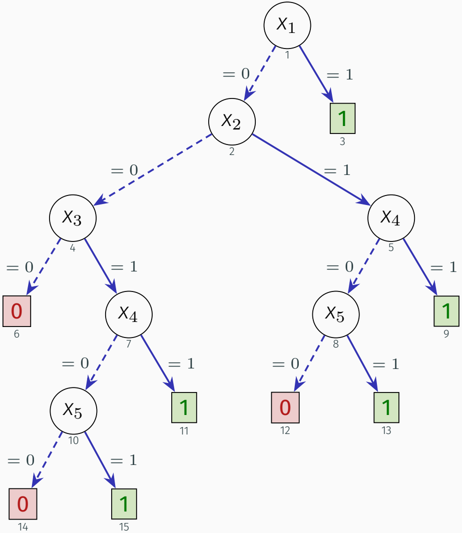

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x_3	x_5	x_1	x_2	x_4	$\kappa(\mathbf{x})$
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1	1	0	0	1	1
1	1	0	1	0	1
1	1	0	1	1	1
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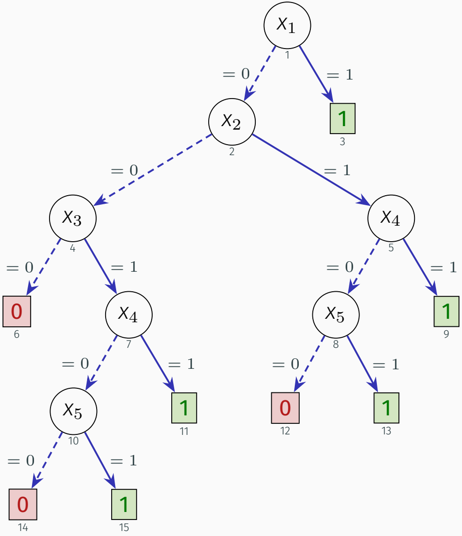

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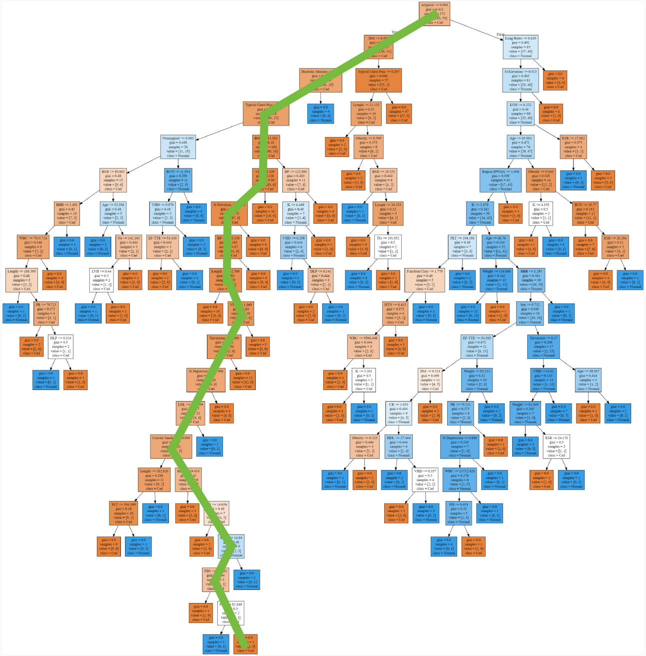
$\therefore$ one AXp is $\{3, 5\}$
Compare with $\{1, 2, 3, 4, 5\}$...

Are *interpretable* models really interpretable? – large DTs

[GZM20]



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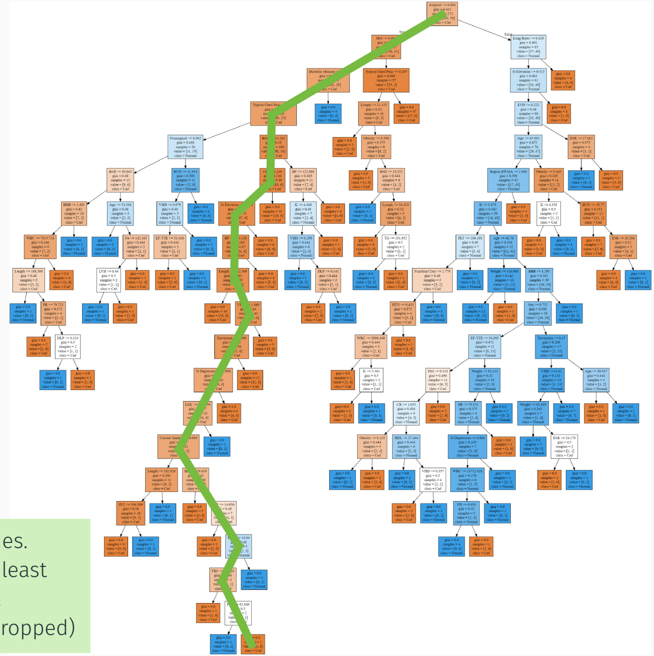

Path with 19 internal nodes.
By manual inspection, at least 10 literals are redundant!
(And at least 9 features dropped)

Are *interpretable* models really interpretable? – large DTs

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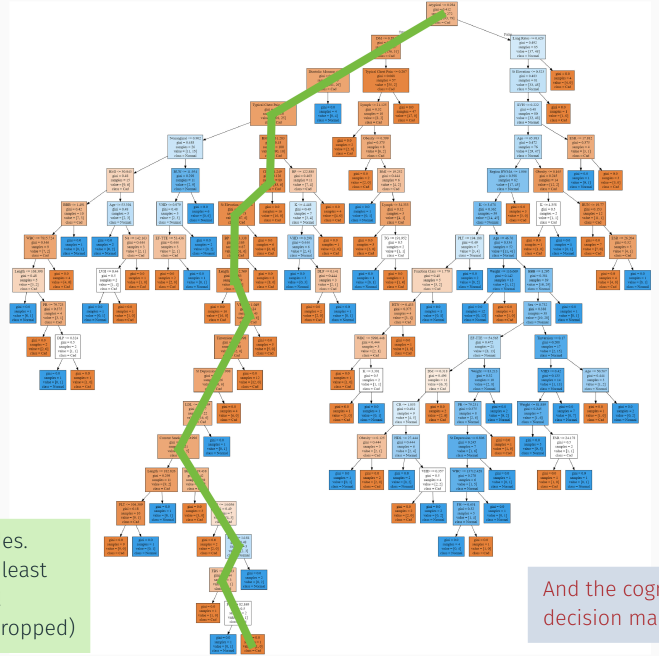

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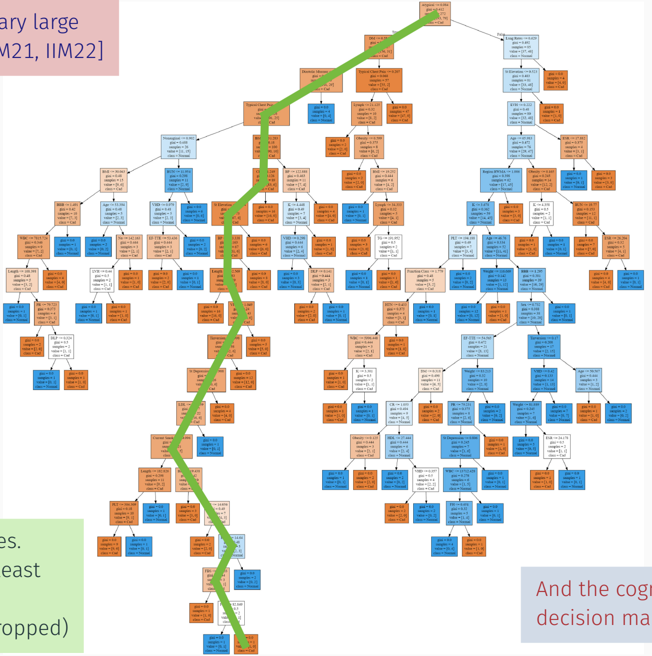
And the cognitive limits of human decision makers are well-known [Mil56]

Are *interpretable* models really interpretable? – large DTs

Redundancy can be arbitrary large on path length [IIM20, HIIM21, IIM22]



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Are *interpretable* models really interpretable? – arbitrary redundancy [IIM20, HIIM21, IIM22]

- Classifier, with $x_1, \ldots, x_m \in \{0, 1\}$:

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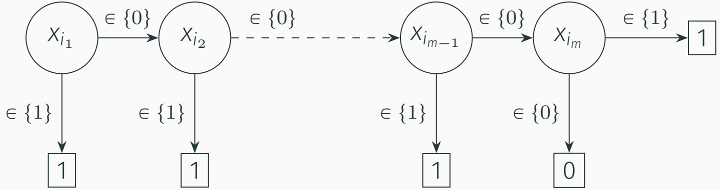

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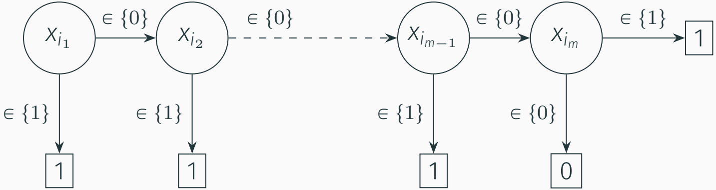

- Point: $(x_{i_1}, x_{i_2}, \ldots, x_{i_{m-1}}, x_{i_m}) = (0, 0, \ldots, 0, 1)$, and prediction 1

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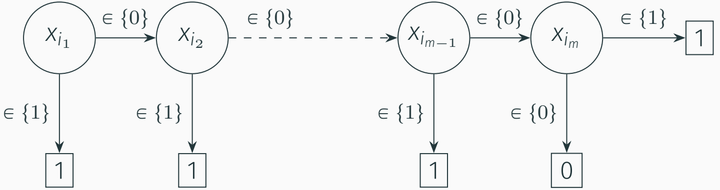

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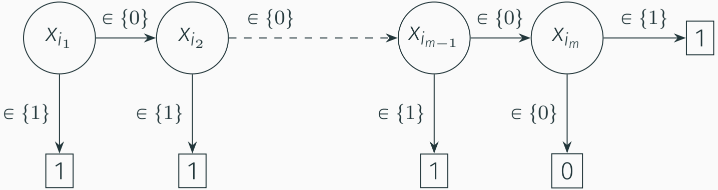

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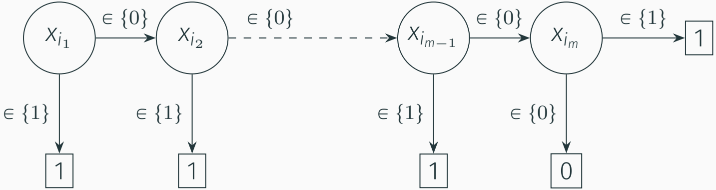

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- **AXp's can be arbitrarily smaller than paths in (optimal) DTs!** [IIM20, IIM22]

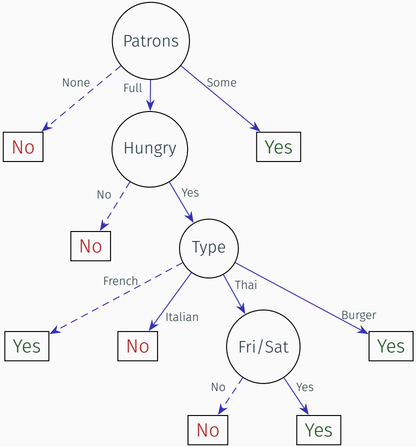
Explanation redundancy in DTs is ubiquitous – published DT examples

[IIM22]

DT Ref	D	#N	#P	%R	%C	%m	%M	%avg
[Alp14, Ch. 09, Fig. 9.1]	2	5	3	33	25	50	50	50
[Alp16, Ch. 03, Fig. 3.2]	2	5	3	33	25	50	50	50
[Bra20, Ch. 01, Fig. 1.3]	4	9	5	60	25	25	50	36
[BA97, Figure 1]	3	12	7	14	8	33	33	33
[BBHK10, Ch. 08, Fig. 8.2]	3	7	4	25	12	50	50	50
[BFOS84, Ch. 01, Fig. 1.1]	3	7	4	50	25	33	33	33
[DL01, Ch. 01, Fig. 1.2a]	2	5	3	33	25	33	33	33
[DL01, Ch. 01, Fig. 1.2b]	2	5	3	33	25	33	33	33
[KMND20, Ch. 04, Fig. 4.14]	3	7	4	25	12	50	50	50
[KMND20, Sec. 4.7, Ex. 4]	2	5	3	33	25	50	50	50
[Qui93, Ch. 01, Fig. 1.3]	3	12	7	28	17	33	50	41
[RM08, Ch. 01, Fig. 1.5]	3	9	5	20	12	33	33	33
[RM08, Ch. 01, Fig. 1.4]	3	7	4	50	25	33	33	33
[WFHP17, Ch. 01, Fig. 1.2]	3	7	4	25	12	50	50	50
[VLE+16, Figure 4]	6	39	20	65	63	20	40	33
[Fla12, Ch. 02, Fig. 2.1(right)]	2	5	3	33	25	50	50	50
[Kot13, Figure 1]	3	10	6	33	11	33	33	33
[Mor82, Figure 1]	3	9	5	80	75	33	50	41
[PM17, Ch. 07, Fig. 7.4]	3	7	4	50	25	33	33	33
[RN10, Ch. 18, Fig. 18.6]	4	12	8	25	6	25	33	29
[SB14, Ch. 18, Page 212]	2	5	3	33	25	50	50	50
[Zho12, Ch. 01, Fig. 1.3]	2	5	3	33	25	33	33	33
[BHO09, Figure 1b]	4	13	7	71	50	33	50	36
[Zho21, Ch. 04, Fig. 4.3]	4	14	9	11	2	25	25	25

Many DTs have paths that are not minimal XPs – Russell&Norvig's book

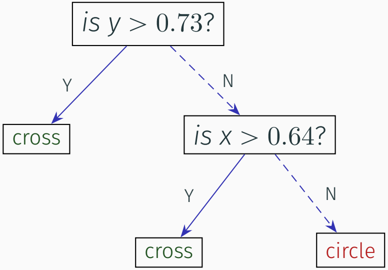
[RN10]



- Explanation for $(\text{P}, \text{H}, \text{T}, \text{W}) = (\text{Full}, \text{Yes}, \text{Thai}, \text{No})$?

Many DTs have paths that are not minimal XPs – Zhou's book

[Zho12]

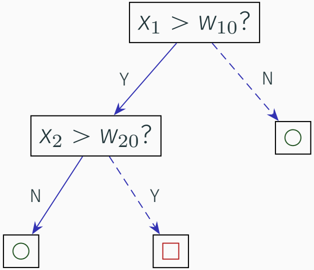


- Explanation for $(x, y) = (1.25, -1.13)$?

Obs: True explanations can be computed for categorical, integer or real-valued features !

Many DTs have paths that are not minimal XPs – Alpaydin's book

[Alp14]

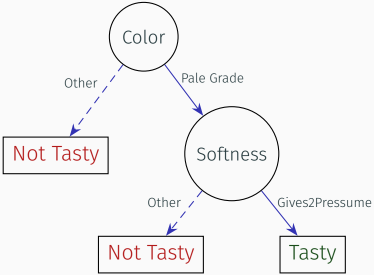


- Explanation for $(x_1, x_2) = (\alpha, \beta)$, with $\alpha > w_{10}$ and $\beta \leqslant w_{20}$?

Obs: True explanations can be computed for categorical, integer or real-valued features !

Many DTs have paths that are not minimal XPs – S.-S.&B.-D.'s book

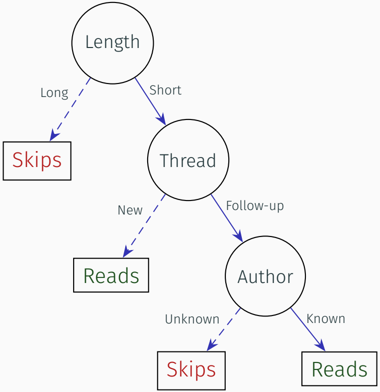
[SB14]



- Explanation for $(\text{color}, \text{softness}) = (\text{Pale Grade}, \text{Other})$?

Many DTs have paths that are not minimal XPs – Poole&Mackworth's book

[PM17]



- Explanation for $(\mathsf{L},\mathsf{T},\mathsf{A}) = (\mathsf{Short},\mathsf{Follow\text{-}Up},\mathsf{Unknown})$?
- Explanation for $(\mathsf{L},\mathsf{T},\mathsf{A}) = (\mathsf{Short},\mathsf{Follow\text{-}Up},\mathsf{Known})$?

Explanation redundancy in DTs is ubiquitous – DTs from datasets

[IIM20, HIIM21, IIM22]

Dataset	(#F	#S)	IAI									ITI								
			D	#N	%A	#P	%R	%C	%m	%M	%avg	D	#N	%A	#P	%R	%C	%m	%M	%avg
adult	(12	6061)	6	83	78	42	33	25	20	40	25	17	509	73	255	75	91	10	66	22
anneal	(38	886)	6	29	99	15	26	16	16	33	21	9	31	100	16	25	4	12	20	16
backache	(32	180)	4	17	72	9	33	39	25	33	30	3	9	91	5	80	87	50	66	54
bank	(19	36293)	6	113	88	57	5	12	16	20	18	19	1467	86	734	69	64	7	63	27
biodegradation	(41	1052)	5	19	65	10	30	1	25	50	33	8	71	76	36	50	8	14	40	21
cancer	(9	449)	6	37	87	19	36	9	20	25	21	5	21	84	11	54	10	25	50	37
car	(6	1728)	6	43	96	22	86	89	20	80	45	11	57	98	29	65	41	16	50	30
colic	(22	357)	6	55	81	28	46	6	16	33	20	4	17	80	9	33	27	25	25	25
compas	(11	1155)	6	77	34	39	17	8	16	20	17	15	183	37	92	66	43	12	60	27
contraceptive	(9	1425)	6	99	49	50	8	2	20	60	37	17	385	48	193	27	32	12	66	21
dermatology	(34	366)	6	33	90	17	23	3	16	33	21	7	17	95	9	22	0	14	20	17
divorce	(54	150)	5	15	90	8	50	19	20	33	24	2	5	96	3	33	16	50	50	50
german	(21	1000)	6	25	61	13	38	10	20	40	29	10	99	72	50	46	13	12	40	22
heart-c	(13	302)	6	43	65	22	36	18	20	33	22	4	15	75	8	87	81	25	50	34
heart-h	(13	293)	6	37	59	19	31	4	20	40	24	8	25	77	13	61	60	20	50	32
kr-vs-kp	(36	3196)	6	49	96	25	80	75	16	60	33	13	67	99	34	79	43	7	70	35
lending	(9	5082)	6	45	73	23	73	80	16	50	25	14	507	65	254	69	80	12	75	25
letter	(16	18668)	6	127	58	64	1	0	20	20	20	46	4857	68	2429	6	7	6	25	9
lymphography	(18	148)	6	61	76	31	35	25	16	33	21	6	21	86	11	9	0	16	16	16
mortality	(118	13442)	6	111	74	56	8	14	16	20	17	26	865	76	433	61	61	7	54	19
mushroom	(22	8124)	6	39	100	20	80	44	16	33	24	5	23	100	12	50	31	20	40	25
pendigits	(16	10992)	6	121	88	61	0	0	—	—	—	38	937	85	469	25	86	6	25	11
promoters	(58	106)	1	3	90	2	0	0	—	—	—	3	9	81	5	20	14	33	33	33
recidivism	(15	3998)	6	105	61	53	28	22	16	33	18	15	611	51	306	53	38	9	44	16
seismic_bumps	(18	2578)	6	37	89	19	42	19	20	33	24	8	39	93	20	60	79	20	60	42
shuttle	(9	58000)	6	63	99	32	28	7	20	33	23	23	159	99	80	33	9	14	50	30
soybean	(35	623)	6	63	88	32	9	5	25	25	25	16	71	89	36	22	1	9	12	10
spambase	(57	4210)	6	63	75	32	37	12	16	33	19	15	143	91	72	76	98	7	58	25
spect	(22	228)	6	45	82	23	60	51	20	50	35	6	15	86	8	87	98	50	83	65
splice	(2	3178)	3	7	50	4	0	0	—	—	—	88	177	55	89	0	0	—	—	—

Are *interpretable* models really interpretable? – DLs

[MSI23]

R_1 :	IF	$(x_1 \wedge x_3)$	THEN	$\kappa(\mathbf{x}) = 1$
R_2 :	ELSE IF	$(x_2 \wedge x_4 \wedge x_6)$	THEN	$\kappa(\mathbf{x}) = 0$
R_3 :	ELSE IF	$(\neg x_1 \wedge x_3)$	THEN	$\kappa(\mathbf{x}) = 1$
R_4 :	ELSE IF	$(x_4 \wedge x_6)$	THEN	$\kappa(\mathbf{x}) = 0$
R_5 :	ELSE IF	$(\neg x_1 \wedge \neg x_3)$	THEN	$\kappa(\mathbf{x}) = 1$
R_6 :	ELSE IF	(x_6)	THEN	$\kappa(\mathbf{x}) = 0$
R_{DEF} :	ELSE			$\kappa(\mathbf{x}) = 1$

- Instance: $((0, 1, 0, 1, 0, 1), 0)$, i.e. rule R_2 fires

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  - **Why?**
    - We need 3 (or 1) so that R_1 cannot fire
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  - **Some questions**:
    - Would average human decision maker be able to understand the AXp?
    - Would he/she be able to compute one AXp, by manual inspection?

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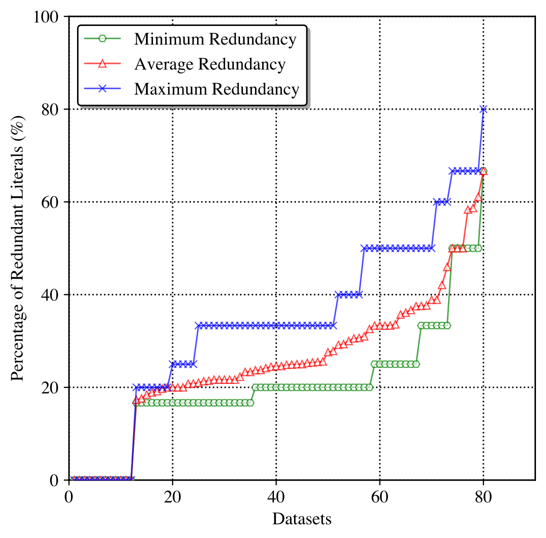
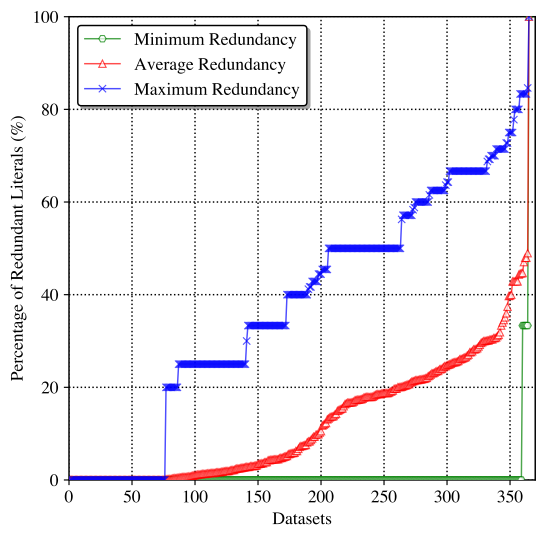
      (BTW, **we have proved that computing one AXp for DLs is computationally hard...**) [IM21, MSI23]



DTs learned with Interpretable AI, max depth 6



DLs learned with CN2

Outline – Unit #03

Explanations for Decision Trees

XAI Queries for DTs

Myth #01: Intrinsic Interpretability

Detour: From Decision Trees to Explained Decision Sets

Explanations for Decision Graphs

Explanations for Monotonic Classifiers

Review examples

From DTs to explained DSs

[HM23a]

- Decision sets raise a number of issues:
  - Overlap: Two rules with different predictions can fire on the same input
  - Incomplete coverage: For some inputs, no rule may fire
    - A default rule defeats the purpose of unordered rules

From DTs to explained DSs

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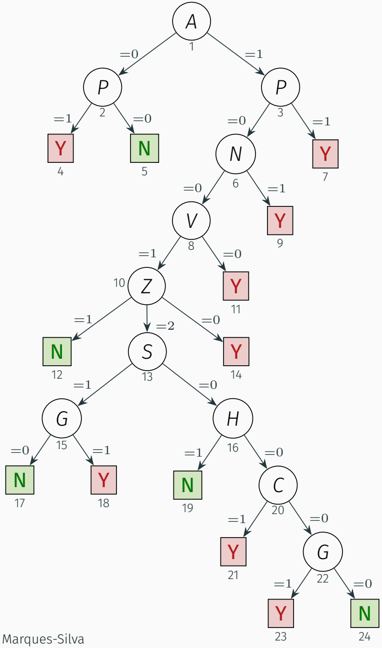
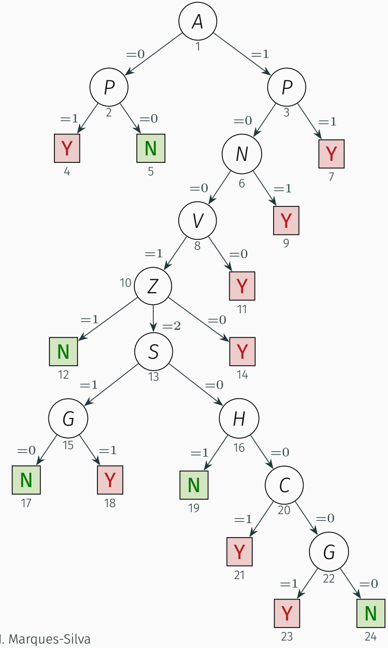
    - A default rule defeats the purpose of unordered rules
  - A DS without overlap and complete coverage computes a classification function
- And explaining DSs is computationally hard...
- One can extract explained DSs from DTs
  - Extract one AXp (viewed as a logic rule) from each path in DT
  - Resulting rules are non-overlapping, and cover feature space

Example



A
1
=0
=1
P
2
P
3
=1
=0
Y
4
N
5
=0
=1
N
6
Y
7
=0
=1
V
8
Y
9
=1
=0
Z
10
Y
11
=1
=2
=0
N
12
S
13
Y
14
=1
=0
G
15
H
16
=0
=1
N
17
Y
18
=1
=0
N
19
C
20
=1
=0
Y
21
G
22
=1
=0
Y
23
N
24

Example



R$_{01}$: IF $[P]$ THEN $\kappa(\cdot) = \mathbf{Y}$

R$_{02}$: IF $[\overline{A} \wedge \overline{P}]$THEN $\kappa(\cdot) = \mathbf{N}$

R$_{03}$: IF $[\overline{P} \wedge \overline{N} \wedge V \wedge Z = 1]$ THEN $\kappa(\cdot) = \mathbf{N}$

R$_{04}$: IF $[\overline{P} \wedge \overline{N} \wedge V \wedge Z = 2 \wedge S \wedge \overline{G}]$ THEN $\kappa(\cdot) = \mathbf{N}$

R$_{05}$: IF $[A \wedge Z = 2 \wedge S \wedge G]$ THEN $\kappa(\cdot) = \mathbf{Y}$

R$_{06}$: IF $[\overline{P} \wedge \overline{N} \wedge V \wedge Z = 2 \wedge \overline{S} \wedge H]$ THEN $\kappa(\cdot) = \mathbf{N}$

R$_{07}$: IF $[A \wedge Z = 2 \wedge \overline{S} \wedge \overline{H} \wedge C]$ THEN $\kappa(\cdot) = \mathbf{Y}$

R$_{08}$: IF $[A \wedge Z = 2 \wedge \overline{H} \wedge G]$ THEN $\kappa(\cdot) = \mathbf{Y}$

R$_{09}$: IF $[\overline{P} \wedge \overline{N} \wedge V \wedge Z = 2 \wedge \overline{C} \wedge \overline{G}]$ THEN $\kappa(\cdot) = \mathbf{N}$

R$_{10}$: IF $[A \wedge Z = 0]$ THEN $\kappa(\cdot) = \mathbf{Y}$

R$_{11}$: IF $[A \wedge \overline{V}]$ THEN $\kappa(\cdot) = \mathbf{Y}$

R$_{12}$: IF $[A \wedge N]$ THEN $\kappa(\cdot) = \mathbf{Y}$

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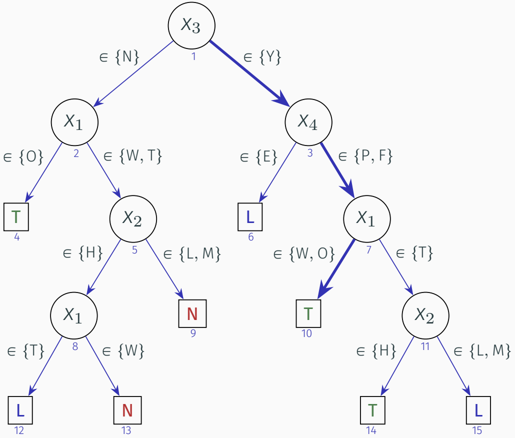
Review examples

Explanation graphs – overview of results

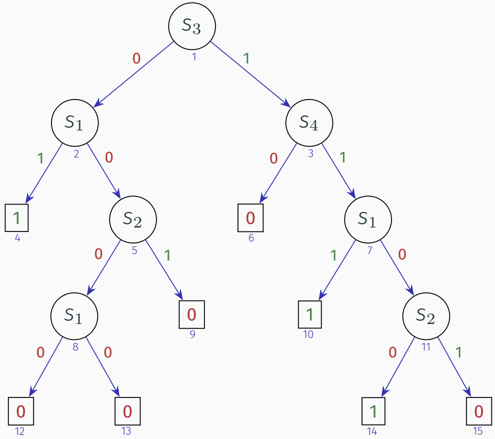
[HIIM21]

- Concept of explanation graph (XpG)
- Explanations of decision trees reducible to XpG's
- Explanations of decision graphs reducible to XpG's
- Explanations of OBDDs reducible to XpG's
- Explanations of OMDDs reducible to XpG's
- Explanations (AXp's and CXp's) of XpG's computed in polynomial time

- DT; point: (O, L, Y, P); prediction T:

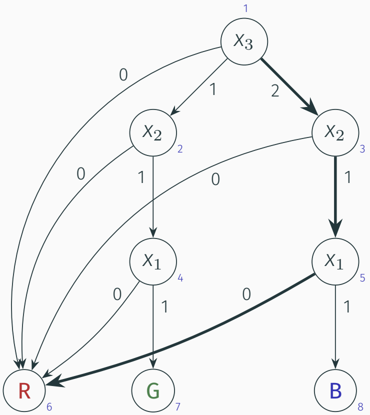


- XpG:

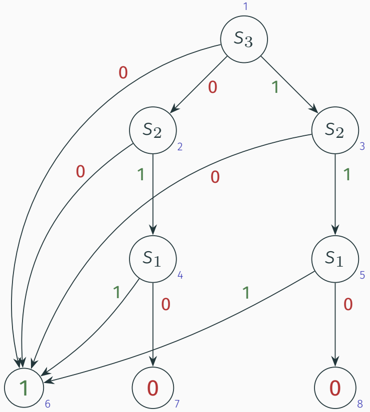


Example of XpG – OMDDs

- OMBBD; point: $(0, 1, 2)$; prediction R:



- XpG:



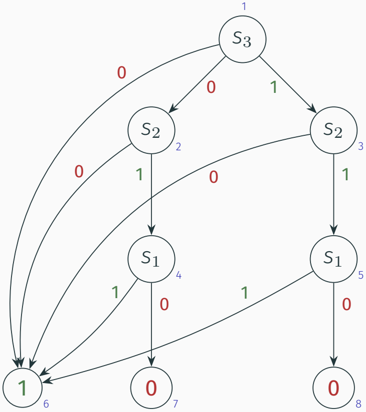
Finding one AXp for XpGs – polynomial time

- Algorithm (with no inconsistent paths):

  $\mathcal{S} \leftarrow \mathcal{F}$

  For each feature i in $\mathcal{F}$

- XpG:



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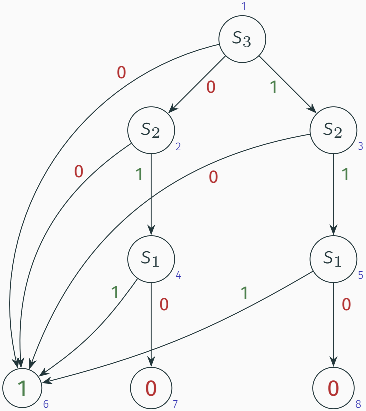

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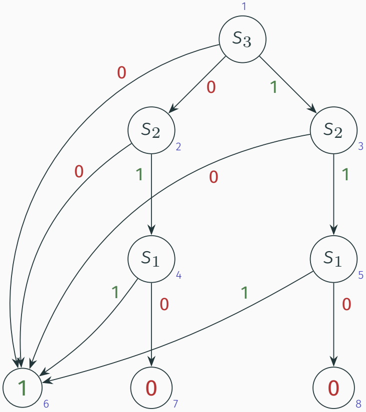

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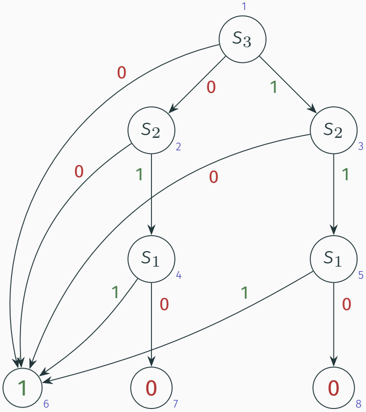

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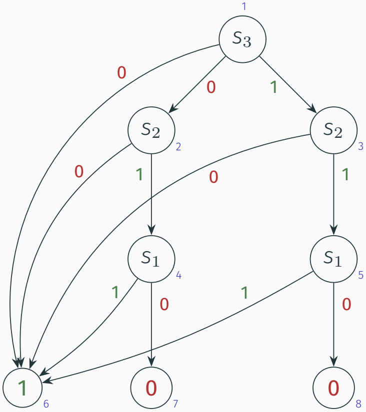
  Return $\mathcal{S}$
- Example:
  - $\mathcal{S} = \{1, 2, 3\}$

- XpG:



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- Example:
  - $\mathcal{S} = \{1, 2, 3\}$
  - Feature 1 cannot be dropped, e.g.

    $s_3 \rightarrow s_2 \rightarrow s_1 \rightarrow 0$

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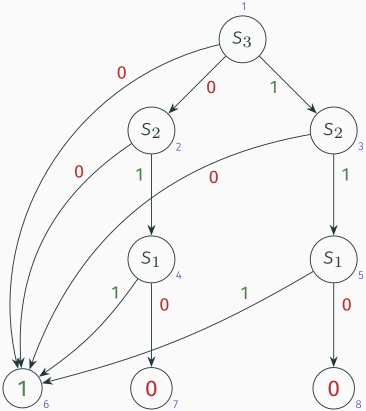

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  Return $\mathcal{S}$

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  - $\mathcal{S} = \{1, 2, 3\}$
  - Feature 1 cannot be dropped, e.g. $s_3 \rightarrow s_2 \rightarrow s_1 \rightarrow 0$
  - Both features 2 and 3 dropped from $\mathcal{S}$

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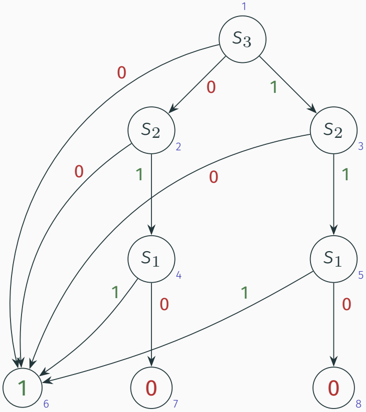

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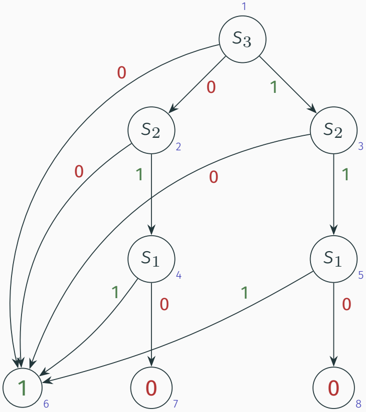
  - Feature 1 cannot be dropped, e.g. $s_3 \rightarrow s_2 \rightarrow s_1 \rightarrow 0$
  - Both features 2 and 3 dropped from $\mathcal{S}$
  - Return $\mathcal{S} = \{1\}$

- XpG:



Example monotonic classifier – $(\mathbf{v}, c) = ((10, 10, 5, 0), A)$

[MGC+21]

Variable	Meaning		Range
$\kappa(\cdot) \triangleq M$	Student grade		$\in \{A, B, C, D, E, F\}$
S	Final score		$\in \{0, \ldots, 10\}$
Feat. id	Feat. var.	Feat. name	Domain
1	Q	Quiz	$\{0, \ldots, 10\}$
2	X	Exam	$\{0, \ldots, 10\}$
3	H	Homework	$\{0, \ldots, 10\}$
4	R	Project	$\{0, \ldots, 10\}$

$$M = \text{ITE}(S \geqslant 9, A, \text{ITE}(S \geqslant 7, B, \text{ITE}(S \geqslant 5, C, \text{ITE}(S \geqslant 4, D, \text{ite}(S \geqslant 2, E, F)))))$$

$$S = \max\left[0.3 \times Q + 0.6 \times X + 0.1 \times H, R\right]$$

Also, $F \preccurlyeq E \preccurlyeq D \preccurlyeq C \preccurlyeq B \preccurlyeq A$

And, $\kappa(\mathbf{x_1}) \preccurlyeq \kappa(\mathbf{x_2})$ if $\mathbf{x}_1 \leqslant \mathbf{x}_2$

Explaining monotonic classifiers

- Instance $(\mathbf{v}, c)$
- Domain for $i \in \mathcal{F}$: $\lambda(i) \leqslant x_i \leqslant \mu(i)$
- Idea: refine lower and upper bounds on the prediction
  - $\mathbf{v}_\mathrm{L}$ and $\mathbf{v}_\mathrm{U}$
- Utilities:
  - FixAttr(i):

    $\mathbf{v}_\mathrm{L} \leftarrow (v_{\mathrm{L}_1}, \ldots, v_i, \ldots, v_{\mathrm{L}_N})$
    $\mathbf{v}_\mathrm{U} \leftarrow (v_{\mathrm{U}_1}, \ldots, v_i, \ldots, v_{\mathrm{U}_N})$
    $(\mathcal{A}, \mathcal{B}) \leftarrow (\mathcal{A} \backslash \{i\}, \mathcal{B} \cup \{i\})$
    return $(\mathbf{v}_\mathrm{L}, \mathbf{v}_\mathrm{U}, \mathcal{A}, \mathcal{B})$

  - FreeAttr(i):

    $\mathbf{v}_\mathrm{L} \leftarrow (v_{\mathrm{L}_1}, \ldots, \lambda(i), \ldots, v_{\mathrm{L}_N})$
    $\mathbf{v}_\mathrm{U} \leftarrow (v_{\mathrm{U}_1}, \ldots, \mu(i), \ldots, v_{\mathrm{U}_N})$
    $(\mathcal{A}, \mathcal{B}) \leftarrow (\mathcal{A} \backslash \{i\}, \mathcal{B} \cup \{i\})$
    return $(\mathbf{v}_\mathrm{L}, \mathbf{v}_\mathrm{U}, \mathcal{A}, \mathcal{B})$

Computing one AXp

$\mathbf{v}_\mathsf{L} \leftarrow (v_1, \ldots, v_N)$

$\mathbf{v}_\mathsf{U} \leftarrow (v_1, \ldots, v_N)$ ▷ Ensures: $\kappa(\mathbf{v}_\mathsf{L}) = \kappa(\mathbf{v}_\mathsf{U})$

$(\mathcal{C}, \mathcal{D}, \mathcal{P}) \leftarrow (\mathcal{F}, \emptyset, \emptyset)$ ▷ $\mathcal{S}$: Some possible seed

for all $i \in \mathcal{S}$ **do**

  $(\mathbf{v}_\mathsf{L}, \mathbf{v}_\mathsf{U}, \mathcal{C}, \mathcal{D}) \leftarrow \text{FreeAttr}(i, \mathbf{v}, \mathbf{v}_\mathsf{L}, \mathbf{v}_\mathsf{U}, \mathcal{C}, \mathcal{D})$ ▷ Require: $\kappa(\mathbf{v}_\mathsf{L}) = \kappa(\mathbf{v}_\mathsf{U})$, given $\mathcal{S}$

for all $i \in \mathcal{F} \setminus \mathcal{S}$ **do** ▷ Loop inv.: $\kappa(\mathbf{v}_\mathsf{L}) = \kappa(\mathbf{v}_\mathsf{U})$

  $(\mathbf{v}_\mathsf{L}, \mathbf{v}_\mathsf{U}, \mathcal{C}, \mathcal{D}) \leftarrow \text{FreeAttr}(i, \mathbf{v}, \mathbf{v}_\mathsf{L}, \mathbf{v}_\mathsf{U}, \mathcal{C}, \mathcal{D})$

  if $\kappa(\mathbf{v}_\mathsf{L}) \neq \kappa(\mathbf{v}_\mathsf{U})$ **then** ▷ If invariant broken, fix it

    $(\mathbf{v}_\mathsf{L}, \mathbf{v}_\mathsf{U}, \mathcal{D}, \mathcal{P}) \leftarrow \text{FixAttr}(i, \mathbf{v}, \mathbf{v}_\mathsf{L}, \mathbf{v}_\mathsf{U}, \mathcal{D}, \mathcal{P})$

return $\mathcal{P}$

- **Obs:** $\mathcal{S} = \emptyset$ for computing a single AXp/CXp

Computing one AXp – example

- $\lambda(i) = 0$ and $\mu(i) = 10$
- $\mathbf{v} = (10, 10, 5, 0)$, with $\kappa(\mathbf{v}) = A$
- Q: find one AXp (CXp is similar)

Feat.	Initial values		Changed values		Predictions		Dec.	Resulting values	
	$\mathbf{v}_L$	$\mathbf{v}_U$	$\mathbf{v}_L$	$\mathbf{v}_U$	$\kappa(\mathbf{v}_L)$	$\kappa(\mathbf{v}_U)$		$\mathbf{v}_L$	$\mathbf{v}_U$
1	(10,10,5,0)	(10,10,5,0)	(0,10,5,0)	(10,10,5,0)	C	A	✓	(10,10,5,0)	(10,10,5,0)
2	(10,10,5,0)	(10,10,5,0)	(10,0,5,0)	(10,10,5,0)	E	A	✓	(10,10,5,0)	(10,10,5,0)
3	(10,10,5,0)	(10,10,5,0)	(10,10,0,0)	(10,10,10,0)	A	A	✗	(10,10,0,0)	(10,10,10,0)
4	(10,10,0,0)	(10,10,10,0)	(10,10,0,0)	(10,10,10,10)	A	A	✗	(10,10,0,0)	(10,10,10,10)

Outline – Unit #03

Explanations for Decision Trees

XAI Queries for DTs

Myth #01: Intrinsic Interpretability

Detour: From Decision Trees to Explained Decision Sets

Explanations for Decision Graphs

Explanations for Monotonic Classifiers

Review examples

Recap computation of (W)AXps/(W)CXps

$$
\begin{aligned}
\text{WAXp}(\mathcal{X}) \quad &:= \quad \forall(\mathbf{x} \in \mathbb{F}). \bigwedge\nolimits_{j \in \mathcal{X}} (x_j = v_j) \rightarrow (\sigma(\mathbf{x})) \\
\text{WCXp}(\mathcal{Y}) \quad &:= \quad \exists(\mathbf{x} \in \mathbb{F}). \bigwedge\nolimits_{j \notin \mathcal{Y}} (x_j = v_j) \land (\neg\sigma(\mathbf{x}))
\end{aligned}
$$

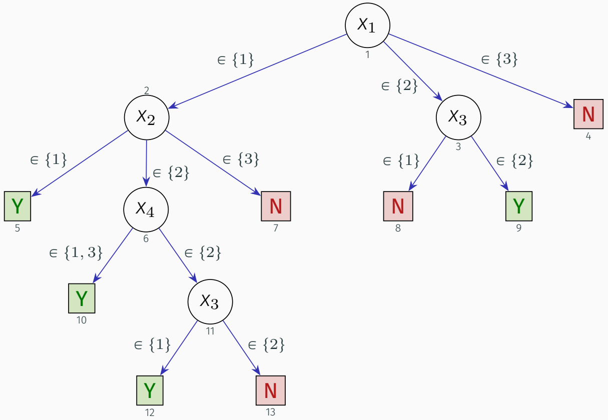
Recap computation of (W)AXps/(W)CXps

$$\begin{aligned} \text{WAXp}(\mathcal{X}) &:= \forall(\mathbf{x} \in \mathbb{F}). \bigwedge\nolimits_{j\in\mathcal{X}}(x_j = v_j) \rightarrow (\sigma(\mathbf{x})) \\ \text{WCXp}(\mathcal{Y}) &:= \exists(\mathbf{x} \in \mathbb{F}). \bigwedge\nolimits_{j\notin\mathcal{Y}}(x_j = v_j) \land (\neg\sigma(\mathbf{x})) \end{aligned}$$

Input: Predicate $\mathbb{P}$, parameterized by $\mathcal{T}$, $\mathcal{M}$
Output: One XP $\mathcal{S}$

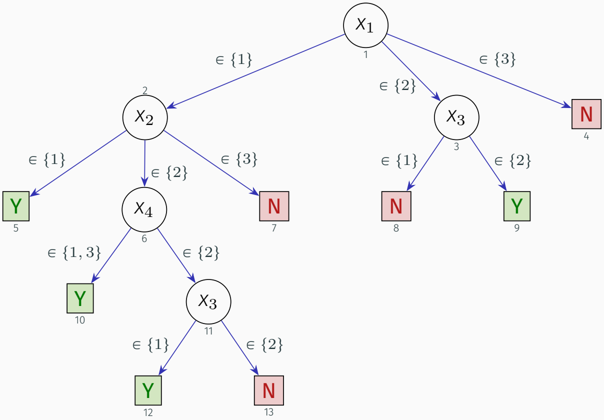
```
procedure oneXP(ℙ)
    S ← F                          ▷ Initialization: ℙ(S) holds
    for i ∈ F do                   ▷ Loop invariant: ℙ(S) holds
        if ℙ(S\{i}) then
            S ← S\{i}              ▷ Update S only if ℙ(S\{i}) holds
    return S                       ▷ Returned set S: ℙ(S) holds
```

- Instance: $(\mathbf{v}, c) = ((1, 2, 1, 2), \text{Y})$



Review exercise – one AXp for example DT

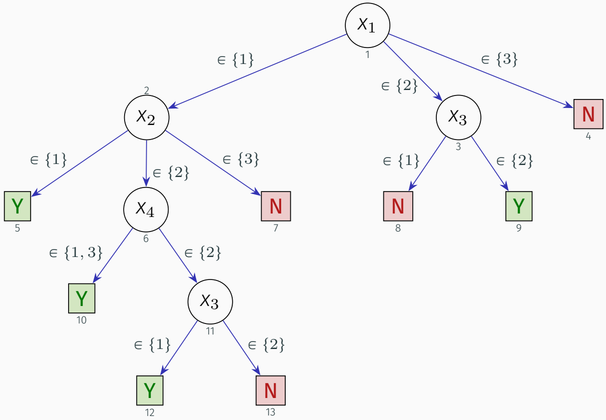
- Instance: $(\mathbf{v}, c) = ((1, 2, 1, 2), \text{Y})$



- Finding on AXp:

Review exercise – one AXp for example DT

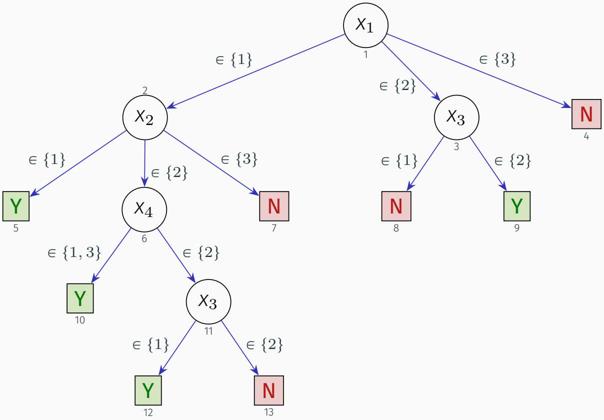
- Instance: $(\mathbf{v}, c) = ((1, 2, 1, 2), \text{Y})$



- Finding on AXp:
  - 1st path inconsistent: $\mathcal{H}_1 = \{3\}$

Review exercise – one AXp for example DT

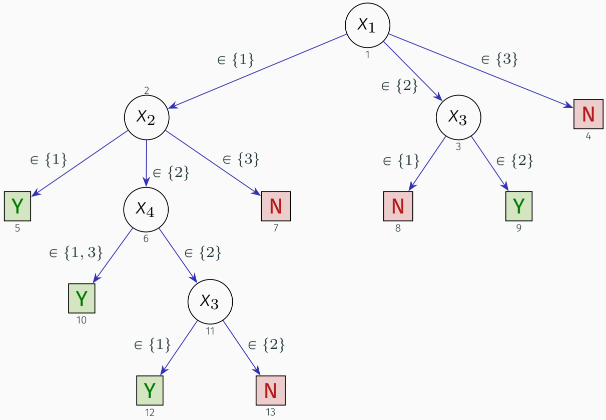
- Instance: $(\mathbf{v}, c) = ((1, 2, 1, 2), \text{Y})$



- Finding on AXp:
  - 1st path inconsistent: $H_1 = \{3\}$
  - 2nd path inconsistent: $H_2 = \{2\}$

Review exercise – one AXp for example DT

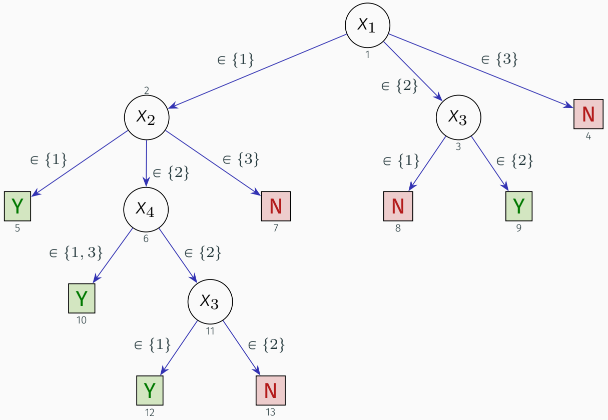
- Instance: $(\mathbf{v}, c) = ((1,2,1,2), \text{Y})$



- Finding on AXp:
  - 1st path inconsistent: $H_1 = \{3\}$
  - 2nd path inconsistent: $H_2 = \{2\}$
  - 3rd path inconsistent: $H_3 = \{1\}$

Review exercise – one AXp for example DT

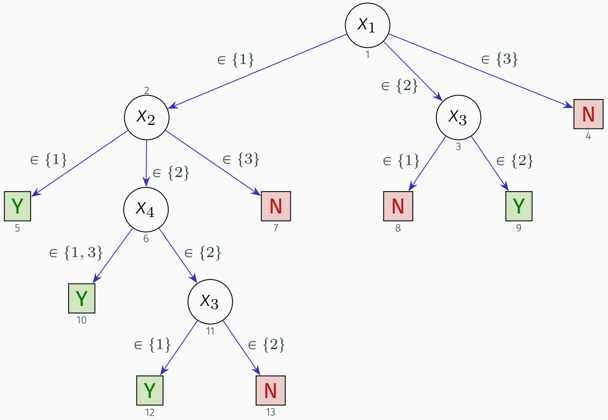
- Instance: $(\mathbf{v}, c) = ((1, 2, 1, 2), \text{Y})$



- Finding on AXp:
  - 1st path inconsistent: $\mathcal{H}_1 = \{3\}$
  - 2nd path inconsistent: $\mathcal{H}_2 = \{2\}$
  - 3rd path inconsistent: $\mathcal{H}_3 = \{1\}$
  - 4th path inconsistent: $\mathcal{H}_4 = \{1\}$

Review exercise – one AXp for example DT

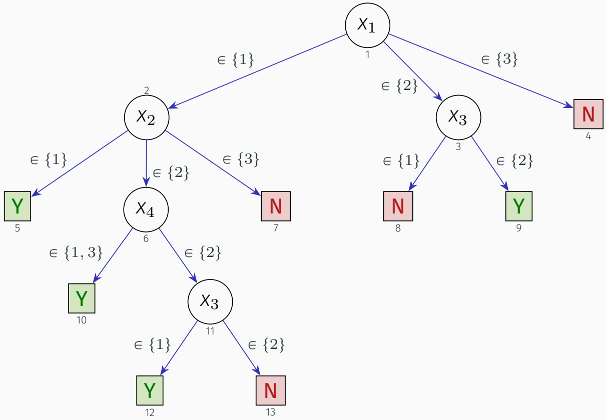
- Instance: $(\mathbf{v}, c) = ((1, 2, 1, 2), \text{Y})$



- Finding on AXp:
  - 1st path inconsistent: $H_1 = \{3\}$
  - 2nd path inconsistent: $H_2 = \{2\}$
  - 3rd path inconsistent: $H_3 = \{1\}$
  - 4th path inconsistent: $H_4 = \{1\}$
- AXp is MHS of H_j sets: $\{1, 2, 3\}$

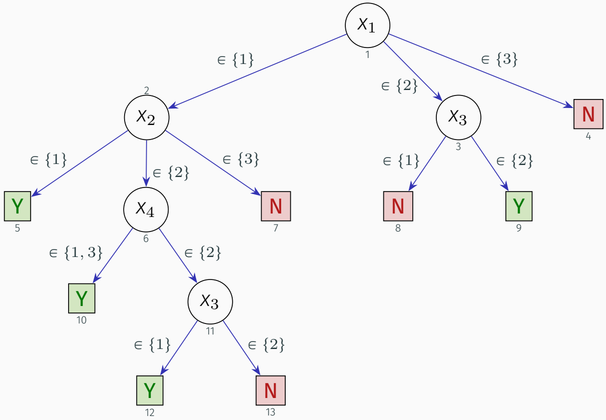
Review exercise – all CXps & AXps for example DT

- Instance: $(\mathbf{v}, c) = ((1, 2, 1, 2), \text{Y})$



Review exercise – all CXps & AXps for example DT

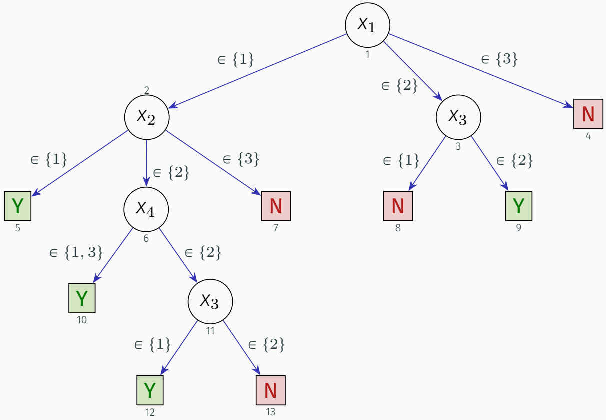
- Instance: $(\mathbf{v}, c) = ((1, 2, 1, 2), \text{Y})$



- Finding CXps:

Review exercise – all CXps & AXps for example DT

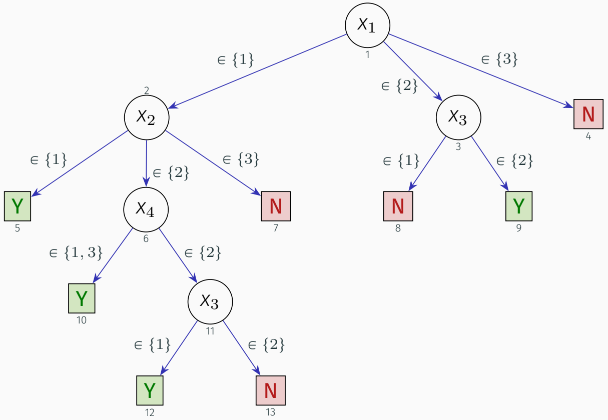
- Instance: $(\mathbf{v}, c) = ((1, 2, 1, 2), \text{Y})$



- Finding CXps:
  - 1st path: $I_1 = \{3\}$

Review exercise – all CXps & AXps for example DT

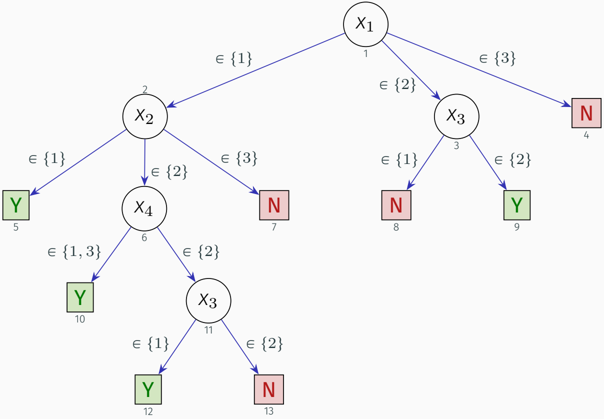
- Instance: $(\mathbf{v}, c) = ((1, 2, 1, 2), \text{Y})$



- Finding CXps:
  - 1st path: $I_1 = \{3\}$
  - 2nd path: $I_2 = \{2\}$

Review exercise – all CXps & AXps for example DT

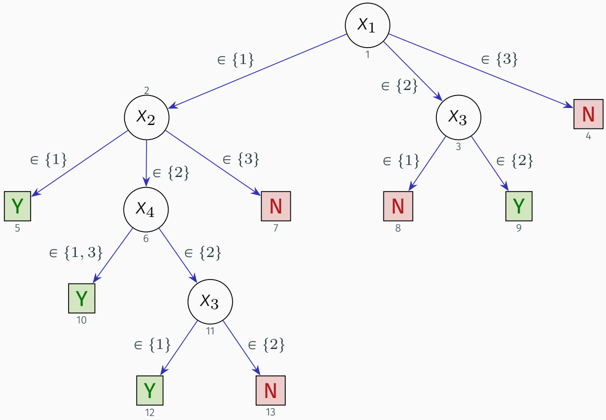
- Instance: $(\mathbf{v}, c) = ((1, 2, 1, 2), \text{Y})$



- Finding CXps:
  - 1st path: $I_1 = \{3\}$
  - 2nd path: $I_2 = \{2\}$
  - 3rd path: $I_3 = \{1\}$

Review exercise – all CXps & AXps for example DT

- Instance: $(\mathbf{v}, c) = ((1, 2, 1, 2), \text{Y})$



- Finding CXps:
  - 1st path: $I_1 = \{3\}$
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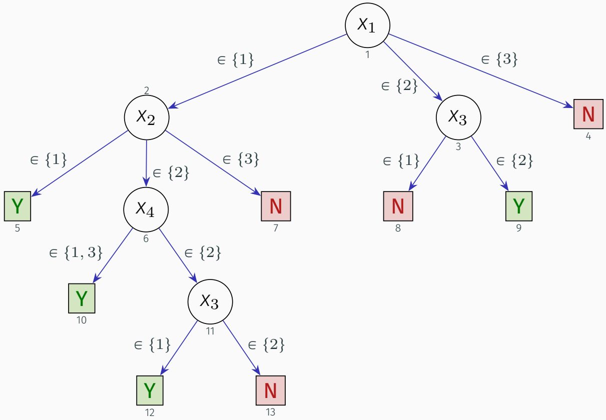
  - 3rd path: $I_3 = \{1\}$
  - 4th path: $I_4 = \{1\}$

Review exercise – all CXps & AXps for example DT

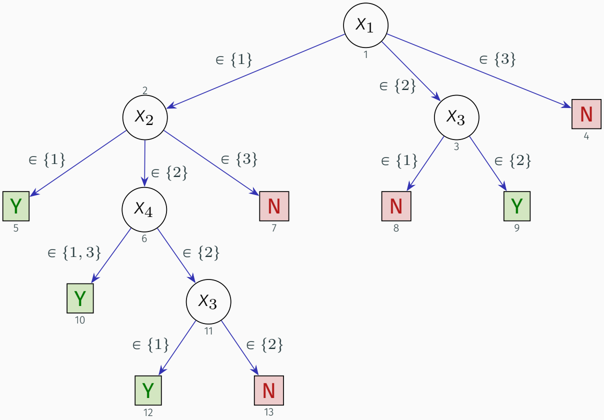
- Instance: $(\mathbf{v}, c) = ((1, 2, 1, 2), \text{Y})$



- Finding CXps:
  - 1st path: $I_1 = \{3\}$
  - 2nd path: $I_2 = \{2\}$
  - 3rd path: $I_3 = \{1\}$
  - 4th path: $I_4 = \{1\}$
  - $\mathcal{L} = \{\{1\}, \{2\}, \{3\}\} = \mathbb{C}$

Review exercise – all CXps & AXps for example DT

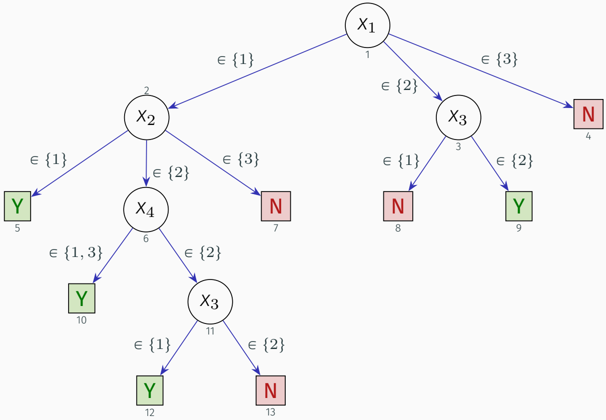
- Instance: $(\mathbf{v}, c) = ((1, 2, 1, 2), \text{Y})$



- Finding CXps:
  - 1st path: $I_1 = \{3\}$
  - 2nd path: $I_2 = \{2\}$
  - 3rd path: $I_3 = \{1\}$
  - 4th path: $I_4 = \{1\}$
  - $\mathcal{L} = \{\{1\}, \{2\}, \{3\}\} = \mathbb{C}$
- Finding AXps:
  (i.e. all MHSes of sets in $\mathbb{C}$

Review exercise – all CXps & AXps for example DT

- Instance: $(\mathbf{v}, c) = ((1,2,1,2), \text{Y})$



- Finding CXps:
  - 1st path: $I_1 = \{3\}$
  - 2nd path: $I_2 = \{2\}$
  - 3rd path: $I_3 = \{1\}$
  - 4th path: $I_4 = \{1\}$
  - $\mathcal{L} = \{\{1\},\{2\},\{3\}\} = \mathbb{C}$
- Finding AXps:
  (i.e. all MHSes of sets in $\mathbb{C}$
  - $\mathbb{A} = \{\{1,2,3\}\}$

Another review exercise – one AXp for example DL

- DL:

R_1 :	IF	$(x_1 \land x_3)$	THEN	$\kappa(\mathbf{x}) = 0$
R_2 :	ELSE IF	$(x_1 \land x_5)$	THEN	$\kappa(\mathbf{x}) = 0$
R_3 :	ELSE IF	$(x_2 \land x_4)$	THEN	$\kappa(\mathbf{x}) = 1$
R_4 :	ELSE IF	$(x_1 \land x_7)$	THEN	$\kappa(\mathbf{x}) = 0$
R_5 :	ELSE IF	$(\neg x_4 \land x_6)$	THEN	$\kappa(\mathbf{x}) = 1$
R_6 :	ELSE IF	$(\neg x_4 \land \neg x_6)$	THEN	$\kappa(\mathbf{x}) = 1$
R_7 :	ELSE IF	$(\neg x_2 \land x_6)$	THEN	$\kappa(\mathbf{x}) = 1$
R_{DEF} :	ELSE			$\kappa(\mathbf{x}) = 0$

Another review exercise – one AXp for example DL

- DL:

$$
\begin{array}{llll}
R_1 : & \text{IF} & (x_1 \land x_3) & \text{THEN} \quad \kappa(\mathbf{x}) = 0 \\
R_2 : & \text{ELSE IF} & (x_1 \land x_5) & \text{THEN} \quad \kappa(\mathbf{x}) = 0 \\
R_3 : & \text{ELSE IF} & (x_2 \land x_4) & \text{THEN} \quad \kappa(\mathbf{x}) = 1 \\
R_4 : & \text{ELSE IF} & (x_1 \land x_7) & \text{THEN} \quad \kappa(\mathbf{x}) = 0 \\
R_5 : & \text{ELSE IF} & (\neg x_4 \land x_6) & \text{THEN} \quad \kappa(\mathbf{x}) = 1 \\
R_6 : & \text{ELSE IF} & (\neg x_4 \land \neg x_6) & \text{THEN} \quad \kappa(\mathbf{x}) = 1 \\
R_7 : & \text{ELSE IF} & (\neg x_2 \land x_6) & \text{THEN} \quad \kappa(\mathbf{x}) = 1 \\
R_{\text{DEF}} : & \text{ELSE} & & \phantom{\text{THEN}} \quad \kappa(\mathbf{x}) = 0
\end{array}
$$

- Instance: $\mathbf{v} = (0, 1, 0, 1, 0, 1, 0)$
  - The prediction is 1, due to R_3

Another review exercise – one AXp for example DL

- DL:

$$
\begin{array}{llccc}
R_1: & \text{IF} & (x_1 \land x_3) & \text{THEN} & \kappa(\mathbf{x}) = 0 \\
R_2: & \text{ELSE IF} & (x_1 \land x_5) & \text{THEN} & \kappa(\mathbf{x}) = 0 \\
R_3: & \text{ELSE IF} & (x_2 \land x_4) & \text{THEN} & \kappa(\mathbf{x}) = 1 \\
R_4: & \text{ELSE IF} & (x_1 \land x_7) & \text{THEN} & \kappa(\mathbf{x}) = 0 \\
R_5: & \text{ELSE IF} & (\neg x_4 \land x_6) & \text{THEN} & \kappa(\mathbf{x}) = 1 \\
R_6: & \text{ELSE IF} & (\neg x_4 \land \neg x_6) & \text{THEN} & \kappa(\mathbf{x}) = 1 \\
R_7: & \text{ELSE IF} & (\neg x_2 \land x_6) & \text{THEN} & \kappa(\mathbf{x}) = 1 \\
R_{\text{DEF}}: & \text{ELSE} & & & \kappa(\mathbf{x}) = 0
\end{array}
$$

- Instance: $\mathbf{v} = (0, 1, 0, 1, 0, 1, 0)$
  - The prediction is 1, due to R_3
- AXp:

Another review exercise – one AXp for example DL

- DL:

R_1 :	IF	$(x_1 \land x_3)$	THEN	$\kappa(\mathbf{x}) = 0$
R_2 :	ELSE IF	$(x_1 \land x_5)$	THEN	$\kappa(\mathbf{x}) = 0$
R_3 :	ELSE IF	$(x_2 \land x_4)$	THEN	$\kappa(\mathbf{x}) = 1$
R_4 :	ELSE IF	$(x_1 \land x_7)$	THEN	$\kappa(\mathbf{x}) = 0$
R_5 :	ELSE IF	$(\neg x_4 \land x_6)$	THEN	$\kappa(\mathbf{x}) = 1$
R_6 :	ELSE IF	$(\neg x_4 \land \neg x_6)$	THEN	$\kappa(\mathbf{x}) = 1$
R_7 :	ELSE IF	$(\neg x_2 \land x_6)$	THEN	$\kappa(\mathbf{x}) = 1$
R_{DEF} :	ELSE			$\kappa(\mathbf{x}) = 0$

- Instance: $\mathbf{v} = (0, 1, 0, 1, 0, 1, 0)$
  - The prediction is 1, due to R_3
- AXp: $\{1, 2\}$

Another review exercise – one AXp for example DL

- DL:

R_1 :	IF	$(x_1 \wedge x_3)$	THEN	$\kappa(\mathbf{x}) = 0$
R_2 :	ELSE IF	$(x_1 \wedge x_5)$	THEN	$\kappa(\mathbf{x}) = 0$
R_3 :	ELSE IF	$(x_2 \wedge x_4)$	THEN	$\kappa(\mathbf{x}) = 1$
R_4 :	ELSE IF	$(x_1 \wedge x_7)$	THEN	$\kappa(\mathbf{x}) = 0$
R_5 :	ELSE IF	$(\neg x_4 \wedge x_6)$	THEN	$\kappa(\mathbf{x}) = 1$
R_6 :	ELSE IF	$(\neg x_4 \wedge \neg x_6)$	THEN	$\kappa(\mathbf{x}) = 1$
R_7 :	ELSE IF	$(\neg x_2 \wedge x_6)$	THEN	$\kappa(\mathbf{x}) = 1$
R_{DEF} :	ELSE			$\kappa(\mathbf{x}) = 0$

- Instance: $\mathbf{v} = (0, 1, 0, 1, 0, 1, 0)$
  - The prediction is 1, due to R_3
- AXp: $\{1, 2\}$
- Quiz: write down the constraints and confirm AXp with SAT solver

Questions?

Lecture 03

Recapitulate second lecture

- Rigorous definitions of abductive and contrastive explanations

Recapitulate second lecture

- Rigorous definitions of abductive and contrastive explanations
- Example algorithm for finding one AXp/CXp

Recapitulate second lecture

- Rigorous definitions of abductive and contrastive explanations
- Example algorithm for finding one AXp/CXp
- Explanations for DTs

Recapitulate second lecture

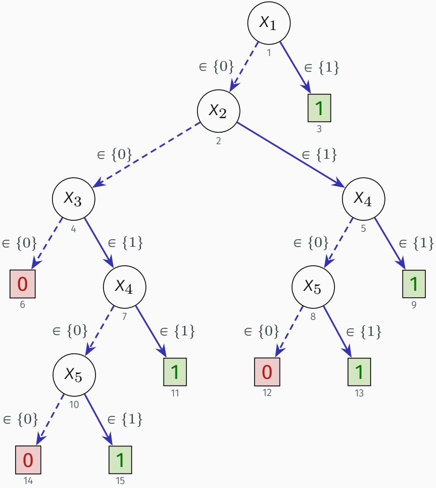
- Rigorous definitions of abductive and contrastive explanations
- Example algorithm for finding one AXp/CXp
- Explanations for DTs
- Explanations for XpGs

Recapitulate second lecture

- Rigorous definitions of abductive and contrastive explanations
- Example algorithm for finding one AXp/CXp
- Explanations for DTs
- Explanations for XpGs
- Explanations for monotonic classifiers

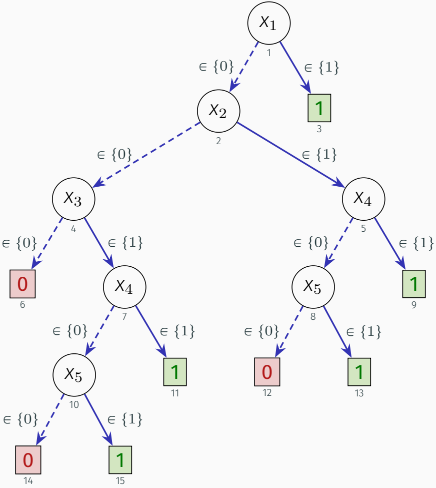
Recap AXps/CXps: DT example

- Instance: $((0, 0, 1, 0, 0), 0)$



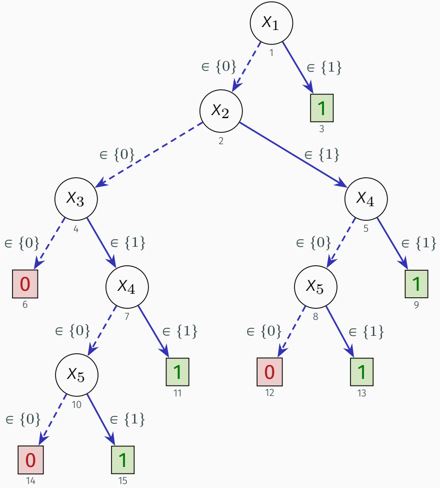
Recap AXps/CXps: DT example

- Instance: $((0,0,1,0,0),0)$
- One AXp: $\{1,4,5\}$



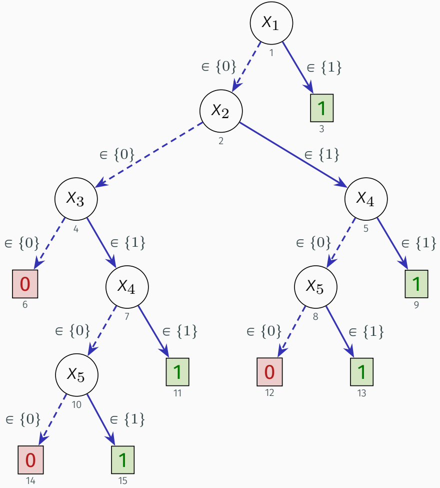
Recap AXps/CXps: DT example

- Instance: $((0, 0, 1, 0, 0), 0)$
- One AXp: $\{1, 4, 5\}$
- All CXps:



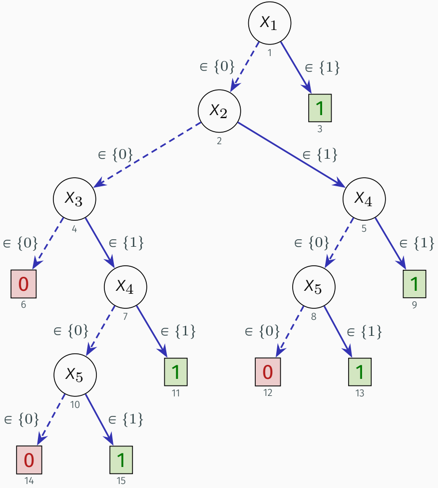
Recap AXps/CXps: DT example

- Instance: $((0,0,1,0,0),0)$
- One AXp: $\{1,4,5\}$
- All CXps:
  - I_1: $\{5\}$



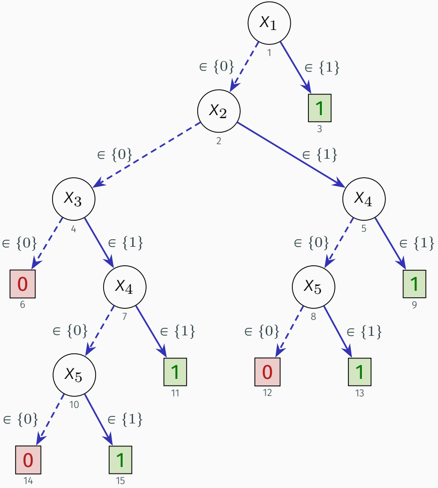
Recap AXps/CXps: DT example

- Instance: $((0,0,1,0,0),0)$
- One AXp: $\{1,4,5\}$
- All CXps:
  - I_1: $\{5\}$
  - I_2: $\{4\}$



Recap AXps/CXps: DT example

- Instance: $((0,0,1,0,0),0)$
- One AXp: $\{1,4,5\}$
- All CXps:
  - I_1: $\{5\}$
  - I_2: $\{4\}$
  - I_3: $\{2,5\}$



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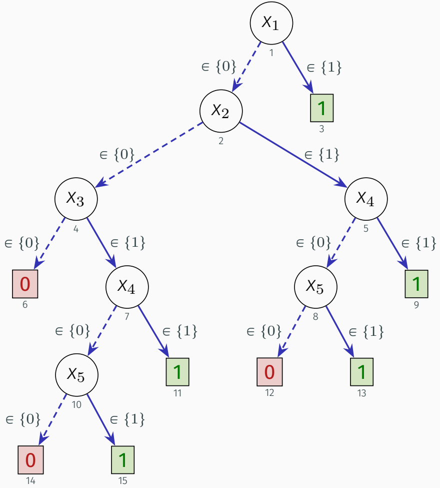

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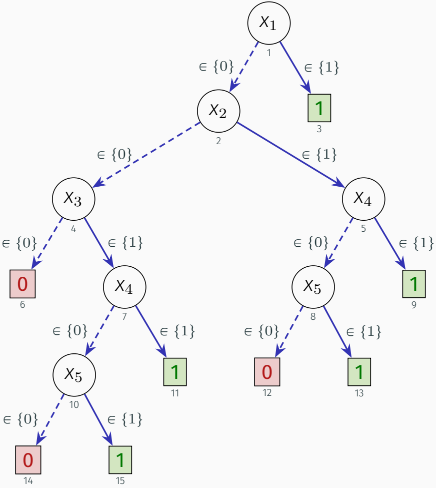

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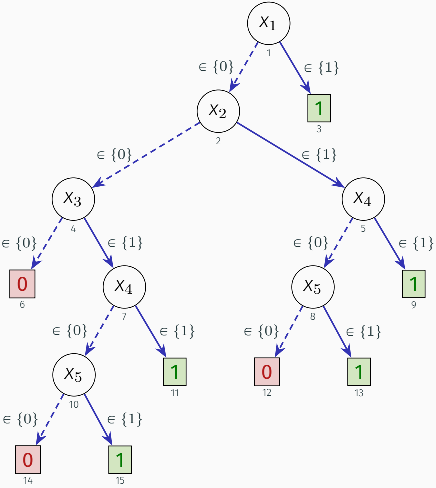
  - $\mathcal{L} = \{\{1\}, \{4\}, \{5\}\}$



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R_1:	IF	$(x_1 = 1)$	THEN	0
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R_3:	ELSE IF	$(x_4 = 1)$	THEN	0
R_{DEF}:	ELSE		THEN	1

Entry	x_1	x_2	x_3	x_4	Rule	$\kappa_1(\mathbf{x})$
00	0	0	0	0	R_{DEF}	1
01	0	0	0	1	R_3	0
02	0	0	0	2	R_{DEF}	1
03	0	0	1	0	R_{DEF}	1
04	0	0	1	1	R_3	0
05	0	0	1	2	R_{DEF}	1
06	0	1	0	0	R_2	1
07	0	1	0	1	R_2	1
08	0	1	0	2	R_2	1
09	0	1	1	0	R_2	1
10	0	1	1	1	R_2	1
11	0	1	1	2	R_2	1
12	1	0	0	0	R_1	0
13	1	0	0	1	R_1	0
14	1	0	0	2	R_1	0
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17	1	0	1	2	R_1	0
18	1	1	0	0	R_1	0
19	1	1	0	1	R_1	0
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- Instance: $(\mathbf{v}, c) = ((0, 0, 1, 2), 1)$
- AXp's: $\{1, 4\}$ (prediction unchanged)
- CXp's:
  - $\{1\}$, by flipping the value of feature 1
  - $\{4\}$, by flipping the value of feature 4
  - But also, $\{\{1\}, \{4\}\}$ by MHS duality

Entry	x_1	x_2	x_3	x_4	Rule	$\kappa_1(\mathbf{x})$
00	0	0	0	0	R_{DEF}	1
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05	0	0	1	2	R_{DEF}	1
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08	0	1	0	2	R_2	1
09	0	1	1	0	R_2	1
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12	1	0	0	0	R_1	0
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16	1	0	1	1	R_1	0
17	1	0	1	2	R_1	0
18	1	1	0	0	R_1	0
19	1	1	0	1	R_1	0
20	1	1	0	2	R_1	0
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22	1	1	1	1	R_1	0
23	1	1	1	2	R_1	0

Plan for this course

- Lecture 01 – units:
  - #01: Foundations
- Lecture 02 – units:
  - #02: Principles of symbolic XAI – feature selection
  - #03: Tractability in symbolic XAI (& myth of interpretability)
- Lecture 03 – units:
  - #04: Intractability in symbolic XAI (& myth of model-agnostic XAI)
  - #05: Explainability queries
- Lecture 04 – units:
  - #06: Advanced topics
- Lecture 05 – units:
  - #07: Principles of symbolic XAI – feature attribution (& myth of Shapley values in XAI)
  - #08: Conclusions & research directions

Some comments...

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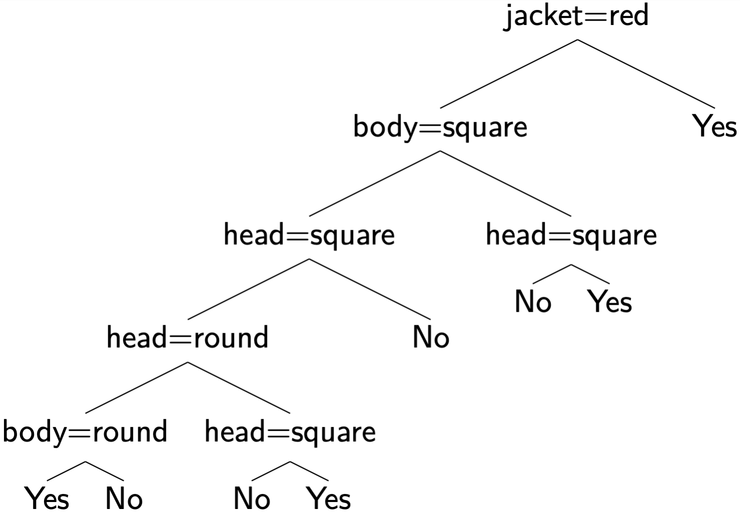
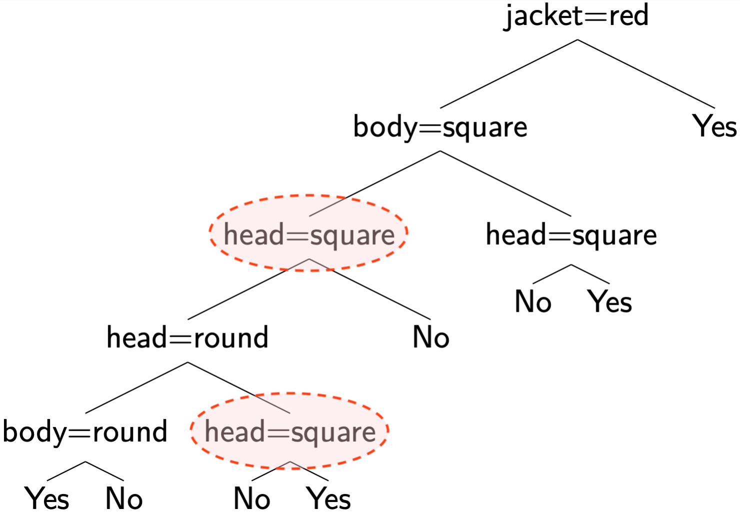
  - More examples next...

Priceless optimal sparse decision trees (OSDT) – & non-optimality!... [HRS19]



Source: Xiyang Hu, Cynthia Rudin, Margo I. Seltzer:
Optimal Sparse Decision Trees.
NeurIPS 2019: 7265-7273



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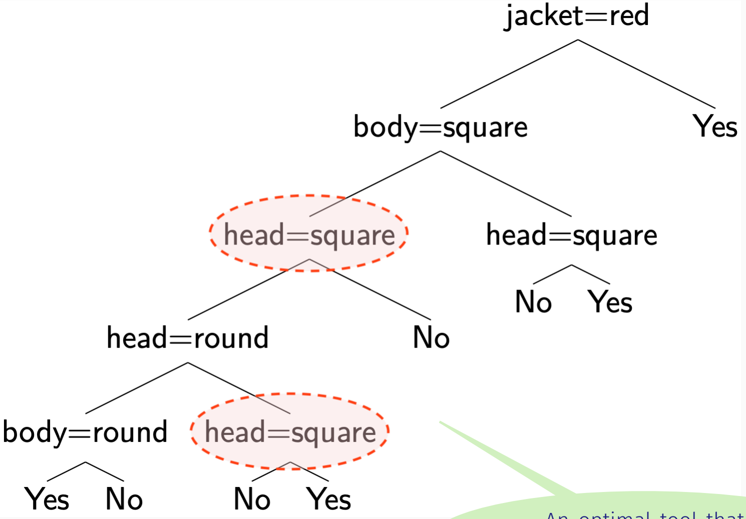

An optimal tool that produces **non-optimal** DTs...!?

Source: Xiyang Hu, Cynthia Rudin, Margo I. Seltzer:
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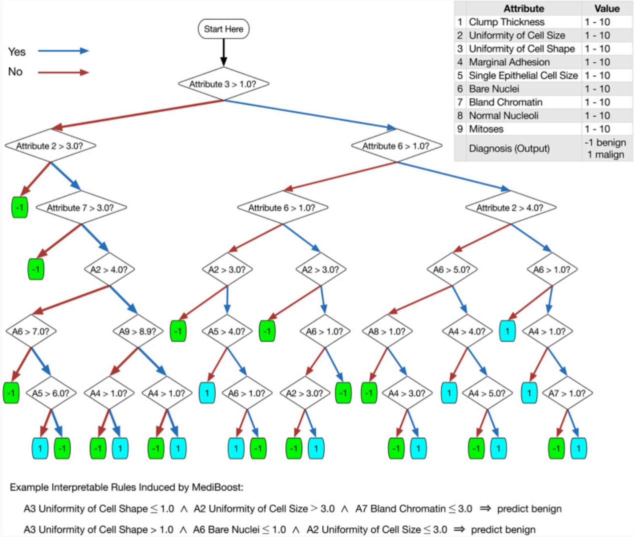
BTW, highly problematic decision trees also in precision medicine...



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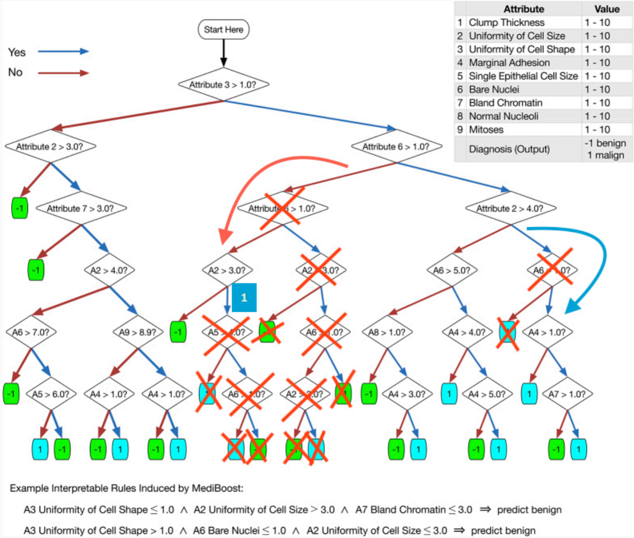

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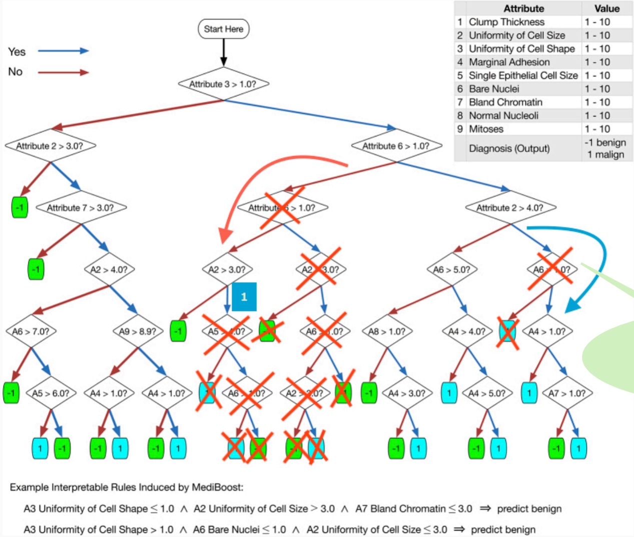

And **massive** path redundancy**!**

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- Previous slides: two examples of obviously buggy DTs
- However, it is relatively simple to implement tree learners
- Can one really trust the operation of more complex ML models, even those subject to extensive testing?
- And how to debug complex ML models if heuristic explanations are also incorrect (more later)?
- **For trustworthy AI, there exists no alternative to rigorous logic-based explanations!**

Unit #04

(Efficient) Intractability in Symbolic XAI

Outline – Unit #04

An encoding for DLs – components

R_1:	IF	(τ_1)	THEN	d_1
R_2:	ELSE IF	(τ_2)	THEN	d_2
		$\cdots$		
R_j:	ELSE IF	(τ_j)	THEN	d_j
		$\cdots$		
R_n:	ELSE IF	(τ_n)	THEN	d_n
R_{DEF}:	ELSE		THEN	d_{n+1}

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- Clauses for encoding ϕ: $\mathfrak{E}_\phi(z_1,\ldots)$, such that $z_1 = 1$ iff $\phi = 1$
- For τ_j: $\mathfrak{E}_{\tau_j}(t_j,\ldots)$
- For $x_i = v_i$: $\mathfrak{E}_{x_i=v_i}(l_i,\ldots)$
- Let $e_j = 1$ iff d_j matches c
- Prediction change with rule up to R_j (with $d_j \neq c$), if $\tau_j \nvDash \bot$ and $\tau_k \vDash \bot$, for $1 \leqslant k < j$, with $e_k = 1$:

$$\left[f_j \leftrightarrow \left(t_j \land \bigwedge_{1\leqslant k<j, e_k=1} \neg t_k\right)\right]$$

An encoding for DLs – components

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- For $x_i = v_i$: $\mathfrak{E}_{x_i=v_i}(l_i, \ldots)$
- Let $e_j = 1$ iff d_j matches c
- Require that at least one f_j, with $e_j = 0$ and $1 \leqslant j \leqslant n$, to be consistent (i.e. some rule up to j with prediction other than c to fire):

$$\left(\bigvee_{1\leqslant j\leqslant n, e_j=0} f_j\right)$$

R_1:	IF	(τ_1)	THEN	d_1
R_2:	ELSE IF	(τ_2)	THEN	d_2
		$\dots$		
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R_n:	ELSE IF	(τ_n)	THEN	d_n
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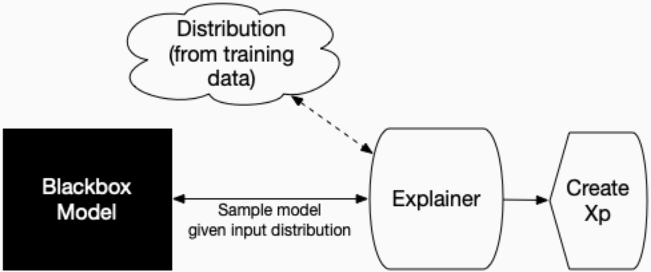
- The set of soft clauses is given by: $\mathcal{S} \triangleq \{(l_i), i = 1, \dots, m\}$
- The set of hard clauses is given by:

$$\mathcal{B} \triangleq \bigwedge_{1\leqslant i\leqslant m} \mathfrak{E}_{x_i=v_i}(l_i, \dots) \land \bigwedge_{1\leqslant j\leqslant n} \mathfrak{E}_{\tau_j}(t_j, \dots) \land$$
$$\bigwedge_{1\leqslant j\leqslant n, e_j=0} \left(f_j \leftrightarrow \left(t_j \land \bigwedge_{1\leqslant k<j, e_k=1} \neg t_k\right)\right) \land \left(\bigvee_{1\leqslant j\leqslant n, e_j=0} f_j\right)$$

- $\mathcal{B} \cup \mathcal{S} \vDash \bot$
    - MUSes are AXp's & MCSes are CXp's

Outline – Unit #04

What is model-agnostic explainability?



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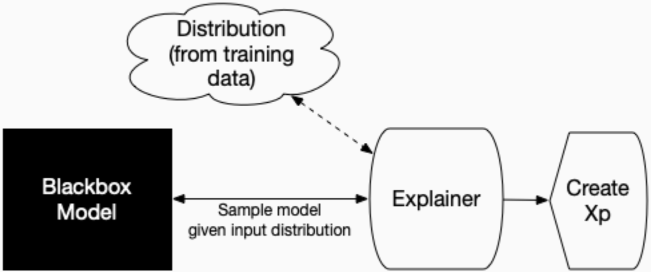
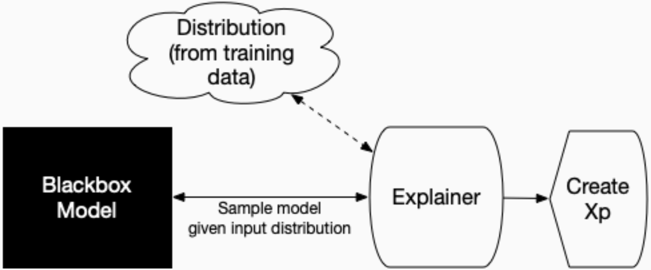


- Wildly popular XAI approach [RSG16, LL17, RSG18]
  - **Feature attribution**: LIME, SHAP, ... [RSG16, LL17]
  - **Feature selection**: Anchors, ... [RSG18]

What is model-agnostic explainability?



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  - **Feature attribution:** LIME, SHAP, ... [RSG16, LL17]
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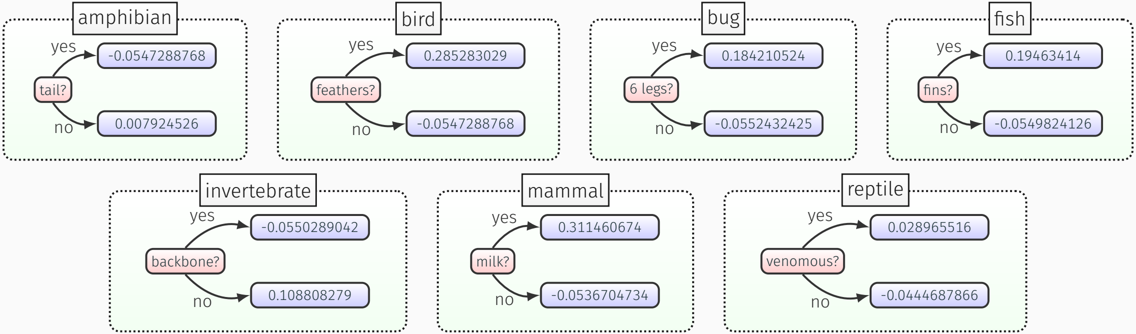
- **Q:** Are model-agnostic explanations rigorous?

Easy to spot problems – BT for zoo dataset

[INM19c, Ign20]



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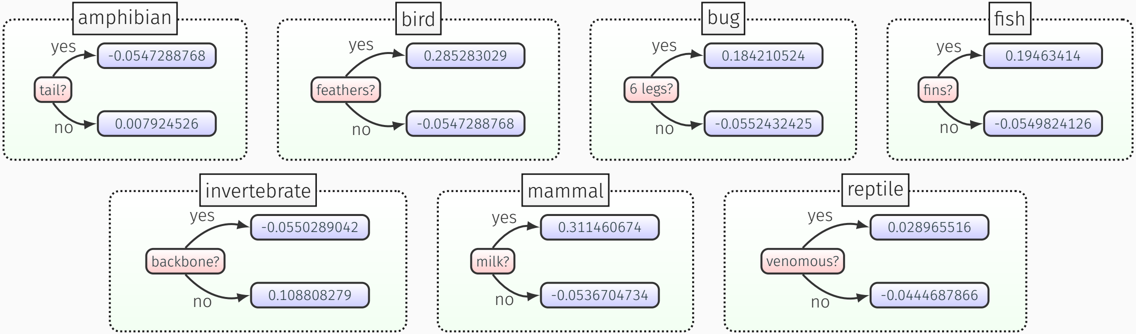

- Example instance:

IF $(\text{animal\_name} = \text{pitviper}) \land \neg\text{hair} \land \neg\text{feathers} \land \text{eggs} \land \neg\text{milk} \land \neg\text{airborne} \land \neg\text{aquatic} \land \text{predator} \land \neg\text{toothed} \land \text{backbone} \land \text{breathes} \land \text{venomous} \land \neg\text{fins} \land (\text{legs} = 0) \land \text{tail} \land \neg\text{domestic} \land \neg\text{catsize}$

THEN $(\text{class} = \text{reptile})$

Easy to spot problems – BT for zoo dataset & Anchor

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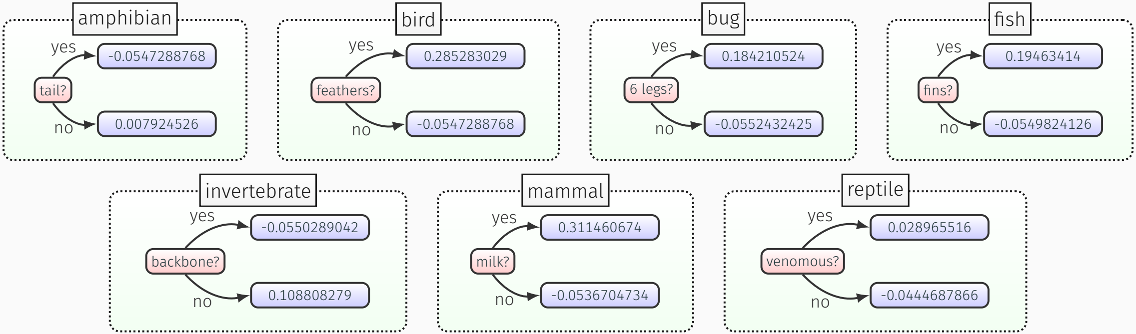

- Example instance (& Anchor picks): [RSG18]

IF	(animal_name = pitviper) $\land$ ¬***hair*** $\land$ ¬feathers $\land$ eggs $\land$ ¬***milk*** $\land$ ¬airborne $\land$ ¬aquatic $\land$ predator $\land$ ¬***toothed*** $\land$ backbone $\land$ breathes $\land$ venomous $\land$ ¬***fins*** $\land$ (legs = 0) $\land$ tail $\land$ ¬domestic $\land$ ¬catsize
THEN	(class = reptile)

Easy to spot problems – BT for zoo dataset & Anchor

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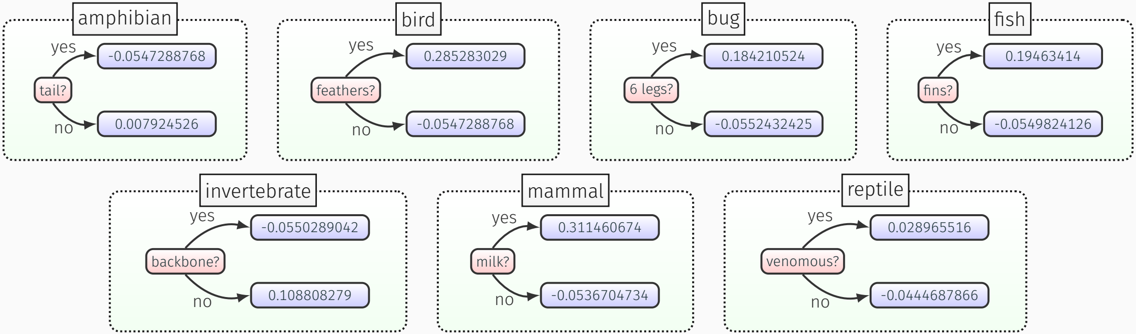


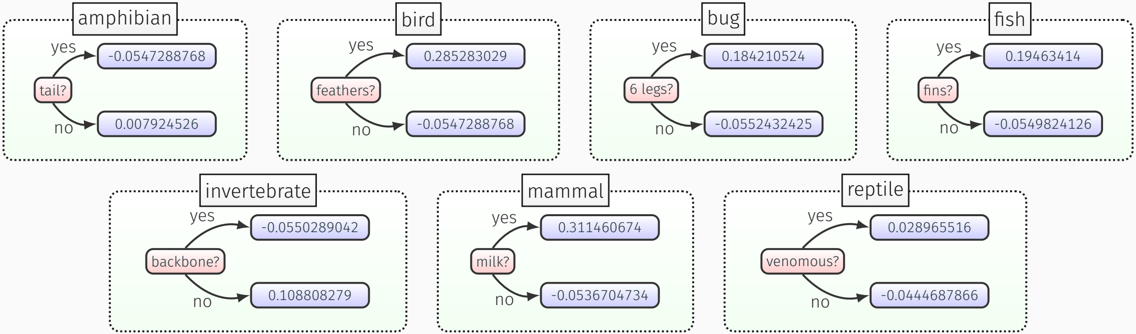
- Explanation obtained with Anchor: [RSG18]

IF $\neg \textit{hair} \wedge \neg \textit{milk} \wedge \neg \textit{toothed} \wedge \neg \textit{fins}$

THEN $(\text{class} = \text{reptile})$

Easy to spot problems – BT for zoo dataset & Anchor

[INM19c, Ign20]



- **But**, explanation **incorrectly "explains"** another instance (from training data!)

IF (animal_name = toad) ∧ ¬***hair*** ∧ ¬feathers ∧ eggs ∧ ¬***milk*** ∧ ¬airborne ∧ ¬aquatic ∧ ¬predator ∧ ¬***toothed*** ∧ backbone ∧ breathes ∧ ¬venomous ∧ ¬***fins*** ∧ (legs = 4) ∧ ¬tail ∧ ¬domestic ∧ ¬catsize

THEN (class = amphibian)

Model-agnostic explainers cannot be trusted

[INM19c]

Incorrect explanations:

Classifier for deciding bank loans

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Two samples: Bessie $:= (v_1, \mathrm{Y})$ and Clive $:= (v_2, \mathrm{N})$

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Two samples: Bessie $:= (v_1, \mathsf{Y})$ and Clive $:= (v_2, \mathsf{N})$

Explanation X: age $= 45$, salary $= 50$K

Model-agnostic explainers cannot be trusted

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Classifier for deciding bank loans

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Explanation X: age $= 45$, salary $=$ 50K

And,

X is consistent with Bessie $:= (\mathbf{v}_1, \mathsf{Y})$

X is consistent with Clive $:= (\mathbf{v}_2, \mathsf{N})$

Incorrect explanations:

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Explanation X: age $= 45$, salary $=$ 50K

And,

- X is consistent with Bessie $\coloneqq (\mathbf{v}_1, \mathsf{Y})$
- X is consistent with Clive $\coloneqq (\mathbf{v}_2, \mathsf{N})$

$\therefore$ different outcomes **&** same explanation **!?**

How to validate model-agnostic explanations

- For feature selection, checking rigor is *easy*

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- Check whether $\mathcal{X}$ is a (rigorous) (W)AXp:
  1. $\mathcal{X}$ is sufficient for prediction:

$$\forall(\mathbf{x} \in \mathbb{F}).\bigwedge_{j\in\mathcal{X}}(x_j = v_j) \rightarrow (\kappa(\mathbf{x}) = c)$$

  2. And, $\mathcal{X}$ is subset-minimal:

$$\forall(t \in \mathcal{X}).\exists(\mathbf{x} \in \mathbb{F}).\bigwedge_{j\in(\mathcal{X}\setminus\{t\})}(x_j = v_j) \rightarrow (\kappa(\mathbf{x}) \neq c)$$

Depending on logic encoding used for classifier, different automated reasoners can be employed

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  Depending on logic encoding used for classifier, different automated reasoners can be employed

- Approach is bounded by scalability of rigorous explanations...

How serious is the lack of rigor of model-agnostic explanations?

- Obs: Lack of rigor of model-agnostic explanations known since 2019 [INM19c, Ign20, YIS+23]

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Dataset	% Incorrect	% Redundant	% Correct
adult	80.5%	1.6%	17.9%
lending	3.0%	0.0%	97.0%
rcdv	99.4%	0.4%	0.2%
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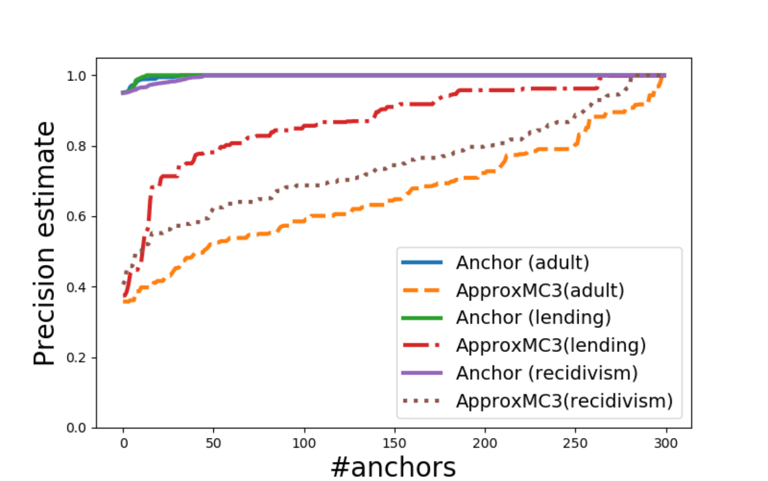
- **Obs:** Results are not positive even if we count how often prediction changes [NSM+19]
  - In this case, BNNs were used, to allow for model counting...
- For feature attribution we proposed different ways of assessing rigor [INM19c, NSM+19, Ign20, YIS+23]

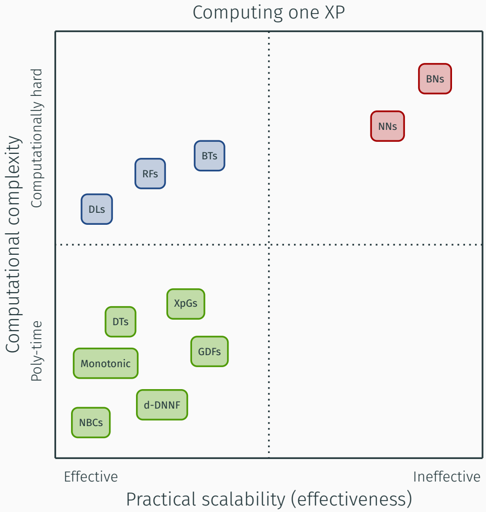
Incorrect explanations are ubiquitous & likely... [NSM+19]



Outline – Unit #04

Efficacy map – progress until 2022

[MI22, Mar22, MS23]



[INM19c, Ign20, IIM20, MGC+20, MGC+21, HIIM21, IMS21, IM21, CM21, HII+22, IISMS22]

- Formal explanations efficient for several families of classifiers
  - Polynomial-time:
    - Naive-Bayes classifiers (NBCs) [MGC+20]
    - Decision trees (DTs) [IIM20, HIIM21]
    - XpG's: DTs, OBDDs, OMDDs, etc. [HIIM21]
    - Monotonic classifiers [MGC+21]
    - Propositional languages (e.g. d-DNNF, ...) [HII+22]
    - Additional results [CM21, HII+22]
  - Comp. hard, but effective (efficient in practice):
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    - Boosted trees (BTs) [INM19c, Ign20, IISMS22]
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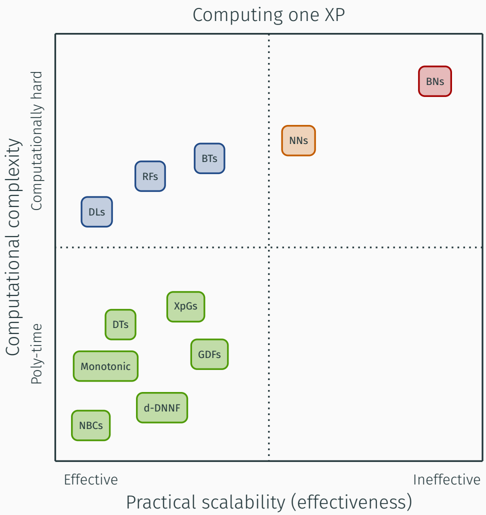

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  - Comp. hard, but some practical scalability:
    - Neural networks (NNs) [HM23b]
  - Comp. hard, and ineffective (hard in practice):
    - Bayesian networks (BNs) [SCD18]

Results for RFs in 2021 (with SAT)

[IMS21]

Dataset	(#F	#C	#I)	RF			CNF		SAT oracle				AXp (**RFxpl**)				Anchor	
				D	#N	%A	#var	#cl	MxS	MxU	#S	#U	Mx	m	avg	%w	avg	%w
ann-thyroid	(21	3	718)	4	2192	98	17854	29230	0.12	0.15	2	18	0.36	0.05	0.13	96	0.32	4
appendicitis	(7	2	43)	6	1920	90	5181	10085	0.02	0.02	4	3	0.05	0.01	0.03	100	0.48	0
banknote	(4	2	138)	5	2772	97	8068	16776	0.01	0.01	2	2	0.03	0.02	0.02	100	0.19	0
biodegradation	(41	2	106	5	4420	88	11007	23842	0.31	1.05	17	22	2.27	0.04	0.29	97	4.07	3
heart-c	(13	2	61)	5	3910	85	5594	11963	0.04	0.02	6	7	0.07	0.01	0.04	100	0.85	0
ionosphere	(34	2	71)	5	2096	87	7174	14406	0.02	0.02	22	11	0.11	0.02	0.03	100	12.43	0
karhunen	(64	10	200)	5	6198	91	36708	70224	1.06	1.41	35	29	14.64	0.65	2.78	100	28.15	0
letter	(16	26	398	8	44304	82	28991	68148	1.97	3.31	8	8	6.91	0.24	1.61	70	2.48	30
magic	(10	2	381)	6	9840	84	29530	66776	0.51	1.84	6	4	2.13	0.07	0.14	99	0.91	1
new-thyroid	(5	3	43)	5	1766	100	17443	28134	0.03	0.01	3	2	0.08	0.03	0.05	100	0.36	0
pendigits	(16	10	220)	6	12004	95	30522	59922	2.40	1.32	10	6	4.11	0.14	0.94	96	3.68	4
ring	(20	2	740	6	6188	89	19114	42362	0.27	0.44	11	9	1.25	0.05	0.25	92	7.25	8
segmentation	(19	7	42)	4	1966	90	21288	35381	0.11	0.17	8	10	0.53	0.11	0.31	100	4.13	0
shuttle	(9	7	116	3	1460	99	18669	29478	0.11	0.08	2	7	0.34	0.05	0.14	99	0.42	1
sonar	(60	2	42)	5	2614	88	9938	20537	0.04	0.06	36	24	0.43	0.04	0.09	100	23.02	0
spectf	(44	2	54)	5	2306	88	6707	13449	0.07	0.06	20	24	0.34	0.02	0.07	100	8.12	0
texture	(40	11	550)	5	5724	87	34293	64187	0.79	0.63	23	17	3.24	0.19	0.93	100	28.13	0
twonorm	(20	2	740	5	6266	94	21198	46901	0.08	0.08	12	8	0.28	0.06	0.10	100	5.73	0
vowel	(13	11	198)	6	10176	90	44523	88696	1.66	2.11	8	5	4.52	0.15	1.15	66	1.67	34
waveform-40	(40	3	500	5	6232	83	30438	58380	0.50	0.86	15	25	7.07	0.11	0.88	100	11.93	0
wpbc	(33	2	78)	5	2432	76	9078	18675	1.00	1.53	20	13	5.33	0.03	0.65	79	3.91	21

Results for NNs in 2019 (with SMT/MILP)

[INM19a]

Dataset			Minimal explanation			Minimum explanation		
			size	SMT (s)	MILP (s)	size	SMT (s)	MILP (s)
australian	(14)	m	1	0.03	0.05	—	—	—
		a	8.79	1.38	0.33	—	—	—
		M	14	17.00	1.43	—	—	—
backache	(32)	m	13	0.13	0.14	—	—	—
		a	19.28	5.08	0.85	—	—	—
		M	26	22.21	2.75	—	—	—
breast-cancer	(9)	m	3	0.02	0.04	3	0.02	0.03
		a	5.15	0.65	0.20	4.86	2.18	0.41
		M	9	6.11	0.41	9	24.80	1.81
cleve	(13)	m	4	0.05	0.07	4	—	0.07
		a	8.62	3.32	0.32	7.89	—	5.14
		M	13	60.74	0.60	13	—	39.06
hepatitis	(19)	m	6	0.02	0.04	4	0.01	0.04
		a	11.42	0.07	0.06	9.39	4.07	2.89
		M	19	0.26	0.20	19	27.05	22.23
voting	(16)	m	3	0.01	0.02	3	0.01	0.02
		a	4.56	0.04	0.13	3.46	0.3	0.25
		M	11	0.10	0.37	11	1.25	1.77
spect	(22)	m	3	0.02	0.02	3	0.02	0.04
		a	7.31	0.13	0.07	6.44	1.61	0.67
		M	20	0.88	0.29	20	8.97	10.73

Results for NNs in 2019 (with SMT/MILP)

[INM19a]

First rigorous approach for **explaining** NNs !

			Minimal explanation			Minimum explanation		
			size	SMT (s)	MILP (s)	size	SMT (s)	MILP (s)
australian	(14)	m	1	0.03	0.05	—	—	—
		a	8.79	1.38	0.33	—	—	—
		M	14	17.00	1.43	—	—	—
backache	(32)	m	13	0.13	0.14	—	—	—
		a	19.28	5.08	0.85	—	—	—
		M	26	22.21	2.75	—	—	—
breast-cancer	(9)	m	3	0.02	0.04	3	0.02	0.03
		a	5.15	0.65	0.20	4.86	2.18	0.41
		M	9	6.11	0.41	9	24.80	1.81
cleve	(13)	m	4	0.05	0.07	4	—	0.07
		a	8.62	3.32	0.32	7.89	—	5.14
		M	13	60.74	0.60	13	—	39.06
hepatitis	(19)	m	6	0.02	0.04	4	0.01	0.04
		a	11.42	0.07	0.06	9.39	4.07	2.89
		M	19	0.26	0.20	19	27.05	22.23
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		M	14	17.00	1.43	—	—	—
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		M	9	6.11	0.41	9	24.80	1.81
cleve	(13)	m	4	0.05	0.07	4	—	0.07
		a	8.62	3.32	0.32	7.89	—	5.14
		M	13	60.74	0.60	13	—	39.06
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		M	11	0.10	0.37	11	1.25	1.77
spect	(22)	m	3	0.02	0.02	3	0.02	0.04
		a	7.31	0.13	0.07	6.44	1.61	0.67
		M	20	0.88	0.29	20	8.97	10.72

Scales to (a few) **tens** of neurons...

Results for NNs in 2023 (using Marabou [KHI+19]) [HM23b]

DNN	points	|AXp|	#Calls	Time	#TO	|AXp|	#Calls	Time	#TO
			$\epsilon = 0.1$				$\epsilon = 0.05$		
ACASXU_1_5	#1	3	5	185.9	0	2	5	113.8	0
	#2	2	5	273.8	0	1	5	33.2	0
	#3	0	5	714.2	0	0	5	4.3	0
ACASXU_3_1	#1	0	5	2219.3	0	0	5	14.2	0
	#2	2	5	4263.5	1	0	5	1853.1	0
	#3	1	5	581.8	0	0	5	355.9	0
ACASXU_3_2	#1	3	5	13739.3	2	1	5	6890.1	1
	#2	3	5	226.4	0	2	5	125.1	0
	#3	2	5	1740.6	0	2	5	173.6	0
ACASXU_3_5	#1	4	5	43.6	0	2	5	59.4	0
	#2	3	5	5039.4	0	2	5	4303.8	1
	#3	2	5	5574.9	1	2	5	2660.3	0
ACASXU_3_6	#1	1	5	6225.0	1	0	5	51.0	0
	#2	3	5	4957.2	1	2	5	1897.3	0
	#3	1	5	196.1	0	1	5	919.2	0
ACASXU_3_7	#1	3	5	6256.2	0	4	5	26.9	0
	#2	4	5	311.3	0	1	5	6958.6	1
	#3	2	5	7756.5	1	1	5	7807.6	1
ACASXU_4_1	#1	2	5	12413.0	2	1	5	5090.5	1
	#2	1	5	5035.1	1	0	5	2335.6	0
	#3	4	5	1237.3	0	4	5	1143.4	0
ACASXU_4_2	#1	4	5	15.9	0	4	5	12.1	0
	#2	3	5	1507.6	0	1	5	111.3	0
	#3	2	5	5641.6	2	0	5	1639.1	0

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	#2	2	5	273.8	0	1	5	33.2	0
	#3	0	5	714.2	0	0	5	4.3	0
ACASXU_3_1	#1	0	5	2219.3	0	0	5	14.2	0
	#2	2	5	4263.5	1	0	5	1853.1	0
	#3	1	5	581.8	0	0	5	355.9	0
ACASXU_3_2	#1	3	5	13739.3	2	1	5	6890.1	1
	#2	3	5	226.4	0	2	5	125.1	0
	#3	2	5	1740.6	0	2	5	173.6	0
ACASXU_3_5	#1	4	5	43.6	0	2	5	59.4	0
	#2	3	5	5039.4	0	2	5	4303.8	1
	#3	2	5	5574.9	1	2	5	2660.3	0
ACASXU_3_6	#1	1	5	6225.0	1	0	5	51.0	0
	#2	3	5	4957.2	1	2	5	1897.3	0
	#3	1	5	196.1	0	1	5	919.2	0
ACASXU_3_7	#1	3	5	6256.2	0	4	5	26.9	0
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Scales to a few **hundred** neurons

More recent results (from 2024)... [IHM+24a]

Model	Deletion							SwiftXplain						
	avgC	nCalls	Len	Mn	Mx	avg	TO	avgC	nCalls	Len	FD%	Mn	Mx	avg
gtsrb-dense	0.06	1024	448	52.0	76.3	63.1	0	0.23	54	447	77.4	10.8	14.0	12.2
gtsrb-convSmall	0.06	1024	309	59.2	82.6	65.1	0	0.22	74	313	39.7	15.1	19.5	16.2
gtsrb-conv	—	—	—	—	—	—	100	96.49	45	174	33.2	3858.7	6427.7	4449.4
mnist-denseSmall	0.28	784	177	190.9	420.3	220.4	0	0.77	111	180	15.5	77.6	104.4	85.1
mnist-dense	0.19	784	231	138.1	179.9	150.6	0	0.75	183	229	11.5	130.1	145.5	136.8
mnist-convSmall	—	—	—	—	—	—	100	98.56	52	116	21.3	4115.2	6858.3	5132.8

More recent results (from 2024)...

[IHM+24a]

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mnist-dense	0.19	784	231	138.1	179.9	150.6	0	0.75	183	229	11.5	130.1	145.5	136.8
mnist-convSmall	—	—	—	—	—	—	100	98.56	52	116	21.3	4115.2	6858.3	5132.8

Scales to **tens of thousands** of neurons!

More recent results (from 2024)...

[IHM+24a]

Model	Deletion							SwiftXplain						
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gtsrb-conv	—	—	—	—	—	—	100	96.49	45	174	33.2	3858.7	6427.7	4449.4
mnist-denseSmall	0.28	784	177	190.9	420.3	220.4	0	0.77	111	180	15.5	77.6	104.4	85.1
mnist-dense	0.19	784	231	138.1	179.9	150.6	0	0.75	183	229	11.5	130.1	145.5	136.8
mnist-convSmall	—	—	—	—	—	—	100	98.56	52	116	21.3	4115.2	6858.3	5132.8

Scales to **tens of thousands** of neurons!

Largest for MNIST: **10142** neurons
Largest for GSTRB: **94308** neurons

Unit #05

Queries in Symbolic XAI

Outline – Unit #05

Enumeration of Explanations

Feature Necessity & Relevancy

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  - There can be too many CXp's...
- Best solution is a MARCO-like algorithm (for enumerating MUSes) [LPMM16]
  - On-demand enumeration of AXp's/CXp's

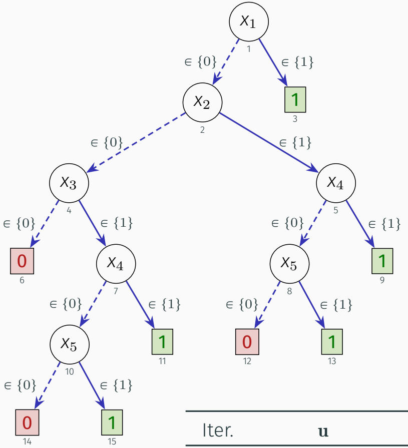
Input: Predicate $\mathbb{P}$, parameterized by $\mathcal{T}$, $\mathcal{M}$
Output: One XP $\mathcal{S}$

procedure oneXP($\mathbb{P}$)
    $\mathcal{S} \leftarrow \mathcal{F}$ ▷ Initialization: $\mathbb{P}(\mathcal{S})$ holds
    for $i \in \mathcal{F}$ **do** ▷ Loop invariant: $\mathbb{P}(\mathcal{S})$ holds
        if $\mathbb{P}(\mathcal{S} \backslash \{i\})$ **then**
            $\mathcal{S} \leftarrow \mathcal{S} \backslash \{i\}$ ▷ Update $\mathcal{S}$ only if $\mathbb{P}(\mathcal{S} \backslash \{i\})$ holds
    return $\mathcal{S}$ ▷ Returned set $\mathcal{S}$: $\mathbb{P}(\mathcal{S})$ holds

Input: Parameters $\mathbb{P}_{\text{axp}}$, $\mathbb{P}_{\text{cxp}}$, $\mathcal{T}$, $\mathcal{F}$, κ, $\mathbf{v}$

$\mathcal{H} \leftarrow \emptyset$ ▷ $\mathcal{H}$ defined on set $U = \{u_1, \ldots, u_m\}$; initially no constraints
repeat
  $(\text{outc}, \mathbf{u}) \leftarrow \text{SAT}(\mathcal{H})$ ▷ Use SAT oracle to pick assignment s.t. known constraints in $\mathcal{H}$
  if outc $=$ **true then**
    $\mathcal{S} \leftarrow \{i \in \mathcal{F} \mid u_i = 0\}$ ▷ $\mathcal{S}$: *fixed* features
    $\mathcal{U} \leftarrow \{i \in \mathcal{F} \mid u_i = 1\}$ ▷ $\mathcal{U}$: *universal* features; $\mathcal{F} = \mathcal{S} \cup \mathcal{U}$
    if $\mathbb{P}_{\text{cxp}}(\mathcal{U}; \mathcal{T}, \mathcal{F}, \kappa, \mathbf{v})$ **then** ▷ $\mathcal{U} = \mathcal{F} \setminus \mathcal{S} \supseteq$ some CXp
      $\mathcal{P} \leftarrow \text{oneXP}(\mathcal{U}; \mathbb{P}_{\text{cxp}}, \mathcal{T}, \mathcal{F}, \kappa, \mathbf{v})$
      reportCXp($\mathcal{P}$)
      $\mathcal{H} \leftarrow \mathcal{H} \cup \{(\vee_{i \in \mathcal{P}} \neg u_i)\}$ ▷ $\mathcal{P} \subseteq \mathcal{U}$: one 1-value variable must be 0 in future iterations
    else ▷ $\mathcal{S} \supseteq$ some AXp
      $\mathcal{P} \leftarrow \text{oneXP}(\mathcal{S}; \mathbb{P}_{\text{axp}}, \mathcal{T}, \mathcal{F}, \kappa, \mathbf{v})$
      reportAXp($\mathcal{P}$)
      $\mathcal{H} \leftarrow \mathcal{H} \cup \{(\vee_{i \in \mathcal{P}} u_i)\}$ ▷ $\mathcal{P} \subseteq \mathcal{S}$: one 0-value variable must be 1 in future iterations
until outc $=$ **false**

DT classifier – example run of enumerator



- Instance: $(\mathbf{v}, c) = ((0, 0, 1, 0, 1), 1)$

x_3	x_5	x_1	x_2	x_4	$\kappa_2(\mathbf{x})$
1	1	0	0	0	1
1	1	0	0	1	1
1	1	0	1	0	1
1	1	0	1	1	1
1	1	1	0	0	1
1	1	1	0	1	1
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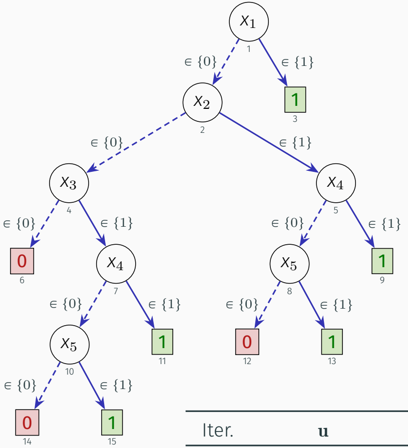
Iter.	$\mathbf{u}$	$\mathcal{S}$	$\mathbb{P}_{\text{cxp}}(\cdot)$	AXp	CXp	Clause	Resulting $\mathcal{H}$
1	$(1, 1, 1, 1, 1)$	$\emptyset$	1	–	$\{3\}$	$(\neg u_3)$	$\{(\neg u_3)\}$
2	$(1, 1, 0, 1, 1)$	$\{3\}$	1	–	$\{5\}$	$(\neg u_5)$	$\{(\neg u_3), (\neg u_5)\}$
3	$(1, 1, 0, 1, 0)$	$\{3, 5\}$	0	$\{3, 5\}$	–	$(u_3 \vee u_5)$	$\{(\neg u_3), (\neg u_5), (u_3 \vee u_5)\}$
5	$[\text{outc} = \textbf{false}]$	–	–	–	–	–	$\{(\neg u_3), (\neg u_5), (u_3 \vee u_5)\}$



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1	$(0,0,0,0,0)$	$\{1,2,3,4,5\}$	0	$\{3,5\}$	–	$(u_3 \vee u_5)$	$\{(u_3 \vee u_5)\}$
2	$(0,0,1,0,0)$	$\{1,2,4,5\}$	1	–	$\{3\}$	$(\neg u_3)$	$\{(u_3 \vee u_5), (\neg u_3)\}$
3	$(0,0,1,0,1)$	$\{1,2,4\}$	1	–	$\{5\}$	$(\neg u_5)$	$\{(u_3 \vee u_5), (\neg u_3), (\neg u_5)\}$
5	[outc = **false**]	–	–	–	–	–	$\{(u_3 \vee u_5), (\neg u_3), (\neg u_5)\}$

DTs admit more efficient algorithms

- Recall:
  - Given instance $(\mathbf{v}, c)$, create set $\mathcal{I}$
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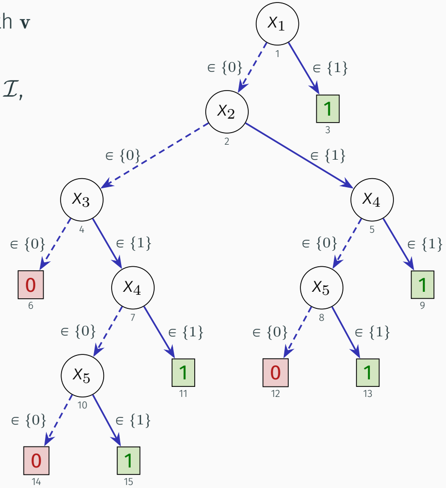
- Example:
  - $I_1 = \{3\}$
  - $I_2 = \{5\}$
  - $I_3 = \{2, 5\}$
  - $\therefore$ keep I_1 an I_2
  - AXp's: MHSes yield $\{\{3, 5\}\}$



Outline – Unit #05

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(Conditioned) Classifier Decision Problem ((C)CDP)

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- Consider instance $(\mathbf{v}, c)$
- Sets of all AXp's & CXp's:

$$\mathbb{A} := \{\mathcal{X} \subseteq \mathcal{F} \,|\, \text{AXp}(\mathcal{X})\}$$

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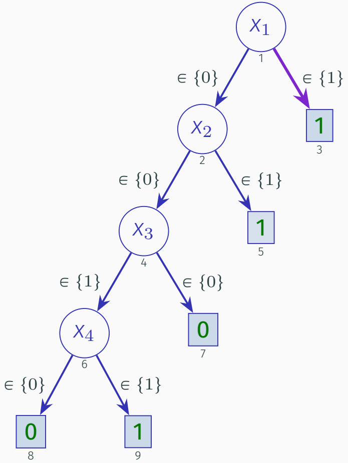
- **Claim #04:** AXp-necessity of $t \in \mathcal{F}$ is in P if t has a domain size which is polynomially-bounded on instance size
  - This holds for any classifier!
  - Let $\mathbf{u}$ be obtained from $\mathbf{v}$ by replacing the constant v_t by some variable $u_t \in \mathcal{D}_t$
  - Feature t is AXp-necessary if $\kappa(\mathbf{u}) \neq \kappa(\mathbf{v})$ for some value $u_t \in \mathcal{D}_t$

An example

- Instance $(\mathbf{v}, c) = ((0,0,0,0),0)$



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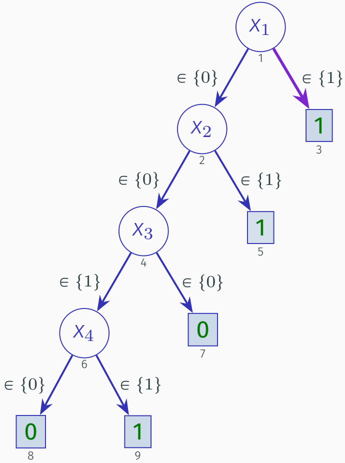
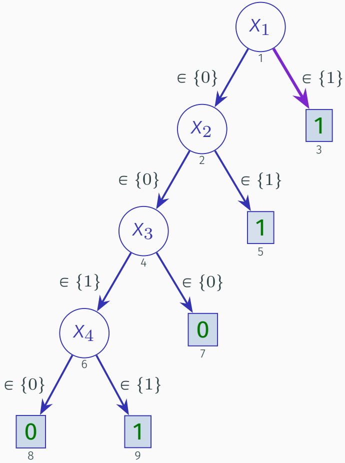


An example

- Instance $(\mathbf{v}, c) = ((0,0,0,0), 0)$
- Is feature 1 AXp-necessary?
  - Does there exist u_1, such that $\kappa(u_1,0,0,0) \neq \kappa(0,0,0,0)$?



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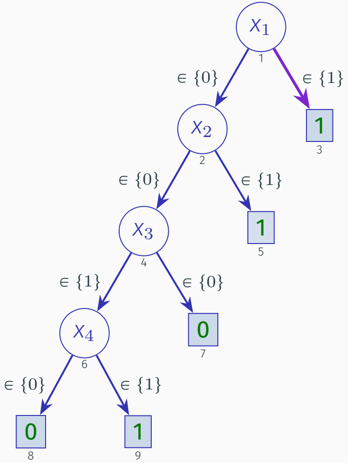
  - Yes! Thus, feature 1 is AXp-necessary



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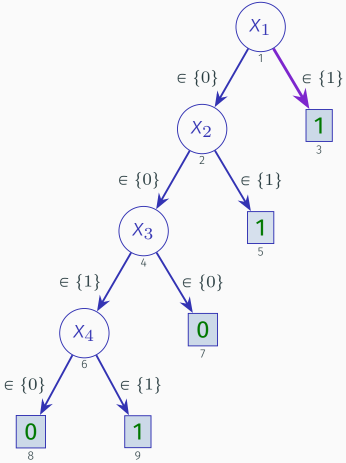
- Is feature 3 AXp-necessary?



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- Is feature 1 AXp-necessary?
  - Does there exist u_1, such that $\kappa(u_1,0,0,0) \neq \kappa(0,0,0,0)$?
  - **Yes!** Thus, feature 1 is AXp-necessary
- Is feature 3 AXp-necessary?
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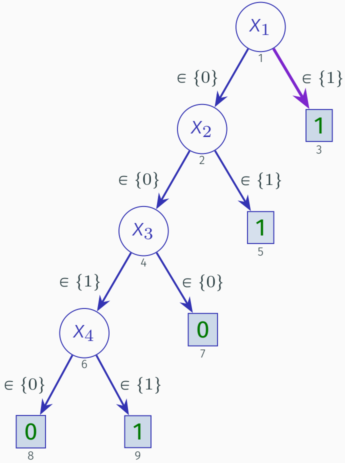

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- Instance $(\mathbf{v}, c) = ((0,0,0,0),0)$
- Is feature 1 AXp-necessary?
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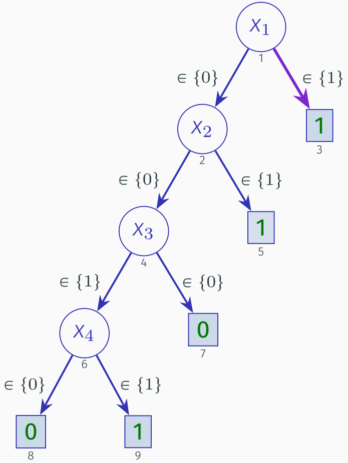
  - Yes! Thus, feature 1 is AXp-necessary
- Is feature 3 AXp-necessary?
  - Does there exist u_3, such that $\kappa(0,0,u_3,0) \neq \kappa(0,0,0,0)$?
  - No! Thus, feature 3 is not AXp-necessary



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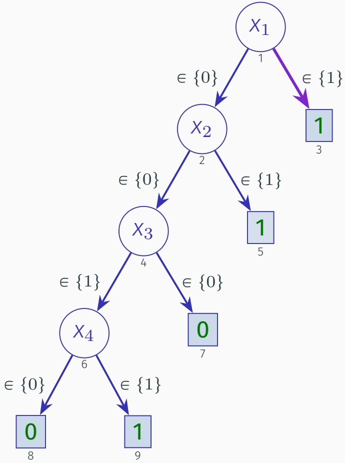
- Is feature 3 AXp-necessary?
  - Does there exist u_3, such that $\kappa(0,0,u_3,0) \neq \kappa(0,0,0,0)$?
  - No! Thus, feature 3 is not AXp-necessary
- Confirmation:
  - CXps:
  - AXps:



An example

- Instance $(\mathbf{v}, c) = ((0,0,0,0),0)$
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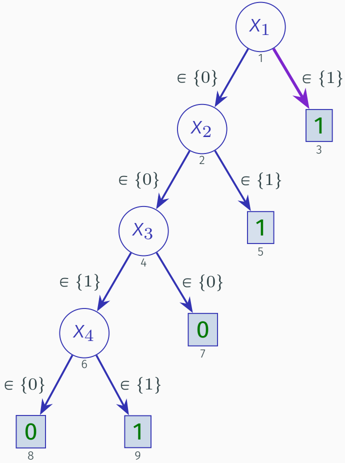

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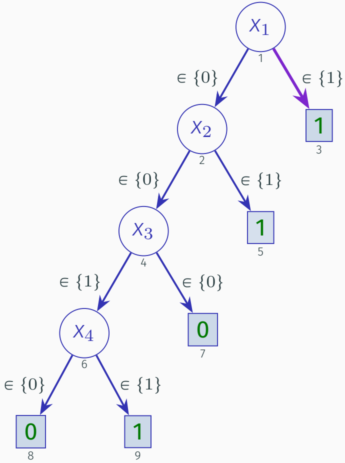

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- Consider instance $(\mathbf{v}, c)$
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- Consider the classifier:

$$\kappa(x_1, x_2, x_3, x_4) = \bigvee_{i=1}^{4} x_i$$

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- Features 1, 2, 3 are irrelevant, since there are not included in any AXp/CXp
  - Obs: **irrelevant features are absolutely unimportant!**

    We could propose some other explanation by adding features 1, 2 or 3 to AXp $\{4\}$, but prediction would remain unchanged for **any** value assigned to those features
    - And we aim for **irreducibility** (**Occam's razor is a mainstay of AI/ML**)

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- General case: best solution is to exploit **abstraction refinement**

Abstraction refinement for feature relevancy

- **Claim**: $\mathcal{X} \subseteq \mathcal{F}$ and $t \in \mathcal{X}$. If WAXp($\mathcal{X}$) holds and WAXp($\mathcal{X} \setminus \{t\}$) does not hold, then **any** AXp $\mathcal{Z} \subseteq \mathcal{X} \subseteq \mathcal{F}$ must contain feature t.

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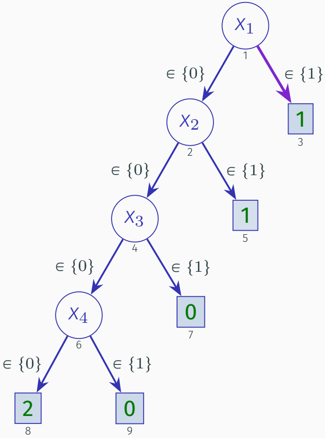
  - Block counterexamples in both cases

A general abstraction refinement algorithm

Input: Instance $\mathbf{v}$, Target Feature t; Feature Set $\mathcal{F}$, Classifier κ

function FRPCGR($\mathbf{v}, t; \mathcal{F}, \kappa$)
  $\mathcal{H} \leftarrow \emptyset$ ▷ $\mathcal{H}$ overapproximates the subsets of $\mathcal{F}$ that do not contain an AXp containing t
  repeat
    $(\mathsf{outc}, \mathbf{s}) \leftarrow \mathsf{SAT}(\mathcal{H}, s_t)$ ▷ **Use SAT oracle to pick candidate WAXp containing** t
    if $\mathsf{outc} = \mathbf{true}$ **then**
      $\mathcal{P} \leftarrow \{i \in \mathcal{F} \mid s_i = 1\}$ ▷ Set $\mathcal{P}$ is the candidate WAXp, and $t \in \mathcal{P}$
      $\mathcal{D} \leftarrow \{i \in \mathcal{F} \mid s_i = 0\}$ ▷ Set $\mathcal{D}$ contains the features not included in $\mathcal{P}$
      if $\neg\mathsf{WAXp}(\mathcal{P})$ **then** ▷ Is $\mathcal{P}$ not a WAXp?
        $\mathcal{H} \leftarrow \mathcal{H} \cup \mathsf{newPosCl}(\mathcal{D}; t, \kappa)$ ▷ $\mathcal{P}$ is *not* a WAXp; must pick some non-picked feature
      else ▷ $\mathcal{P}$ is a WAXp
        if $\neg\mathsf{WAXp}(\mathcal{P}\backslash\{t\})$ **then** ▷ $\mathcal{P}$ without t not a WAXp?
          $\mathsf{reportWeakAXp}(\mathcal{P})$ ▷ Feature t is included in any AXp $\mathcal{X} \subseteq \mathcal{P}$
          return true
        $\mathcal{H} \leftarrow \mathcal{H} \cup \mathsf{newNegCl}(\mathcal{P}; t, \kappa)$ ▷ $\mathsf{WAXp}(\mathcal{P}\backslash\{t\})$ holds; some feature in $\mathcal{P}$ must not be picked
  until $\mathsf{outc} = \mathbf{false}$
  return false ▷ If $\mathcal{H}$ becomes inconsistent, then there is no AXp that contains t

An example: feature relevancy for DT, using abstraction refinement



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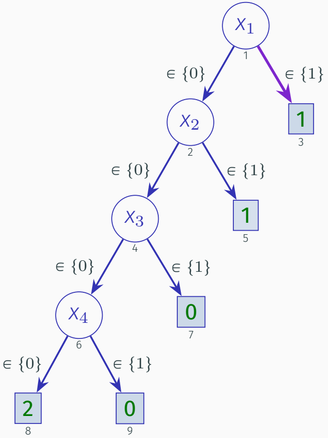

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$(1,1,1,1)$	$\{1,2,3,4\}$	✓	✓	——	$(\neg u_2 \vee \neg u_3 \vee \neg u_4)$
$(1,1,0,1)$	$\{1,2,4\}$	✓	✓	——	$(\neg u_2 \vee \neg u_4)$
$(1,1,0,0)$	$\{1,2\}$	✓	✓	——	$(\neg u_2)$
$(1,0,0,0)$	$\{1\}$	✓	✗	true	——

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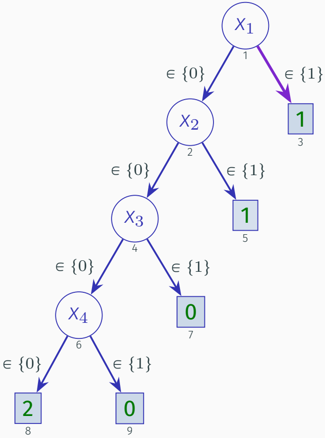

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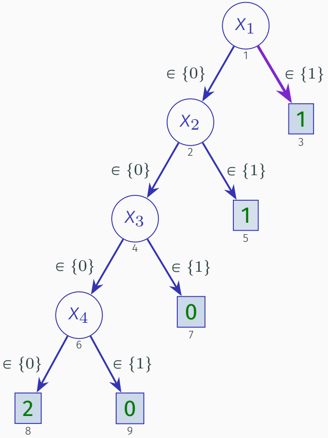

- Instance: $(\mathbf{v}, c) = ((1,1,1,1), 1)$
- Is $t = 4$ relevant?

$t=4$					
$\mathbf{s}$	$\mathcal{P}$	WAXp($\mathcal{P}$)	WAXp($\mathcal{P}\backslash\{t\}$)	Return?	Clause
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$(0,0,0,1)$	$\{4\}$	✗	—	——	$(u_1 \vee u_2 \vee u_3)$
$(0,0,1,1)$	$\{3,4\}$	✗	—	——	$(u_1 \vee u_2)$
[outc = **false**]	——	—	—	**false**	——

Questions?

Lecture 04

Recapitulate third lecture

- Logic encoding for explaining DLs
  - And status of (in)tractability in logic-based XAI
- Query: enumeration of explanations
- Query: feature necessity, AXp & CXp
- Query: feature relevancy

Recap example

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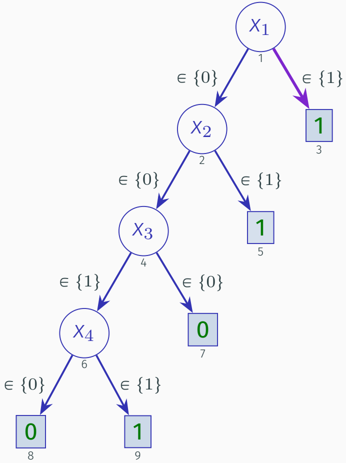

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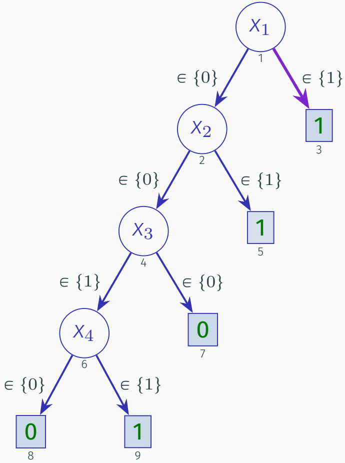

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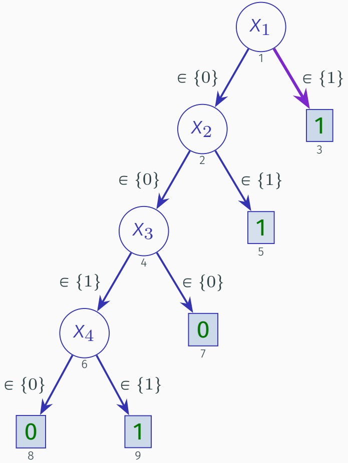

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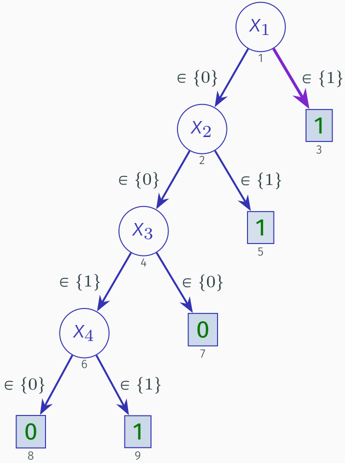
  - Does there exist u_1, such that $\kappa(u_1,0,0,0) \neq \kappa(0,0,0,0)$?
  - Yes! Thus, feature 1 is AXp-necessary (i.e. singleton CXp)



Recap example

- Instance $(\mathbf{v}, c) = ((0,0,0,0),0)$
- Is feature 1 AXp-necessary?
  - Does there exist u_1, such that $\kappa(u_1,0,0,0) \neq \kappa(0,0,0,0)$?
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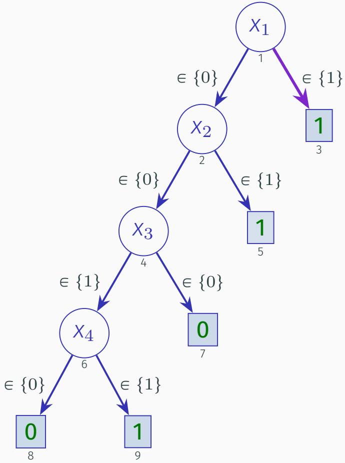

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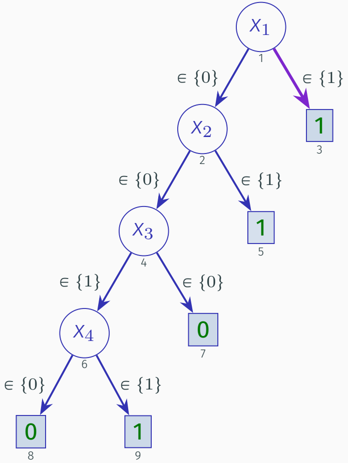

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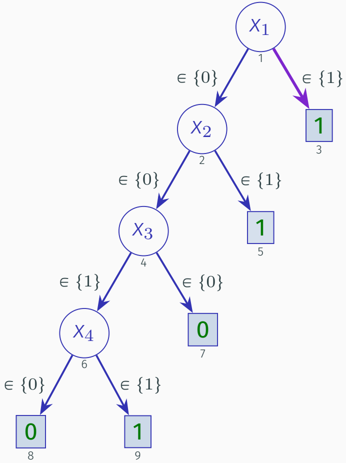
- Is feature 3 AXp-necessary?
  - Does there exist u_3, such that $\kappa(0,0,u_3,0) \neq \kappa(0,0,0,0)$?
  - **No!** Thus, feature 3 is not AXp-necessary



Recap example

- Instance $(\mathbf{v}, c) = ((0, 0, 0, 0), 0)$
- Is feature 1 AXp-necessary?
  - Does there exist u_1, such that $\kappa(u_1, 0, 0, 0) \neq \kappa(0, 0, 0, 0)$?
  - Yes! Thus, feature 1 is AXp-necessary (i.e. singleton CXp)
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- Are there CXp-necessary features?
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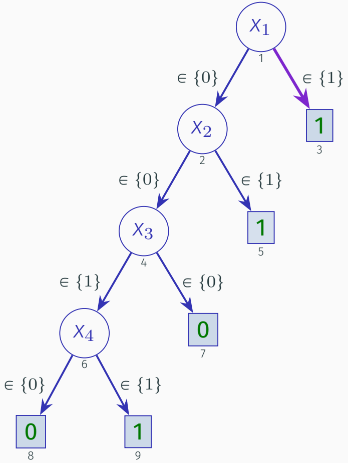

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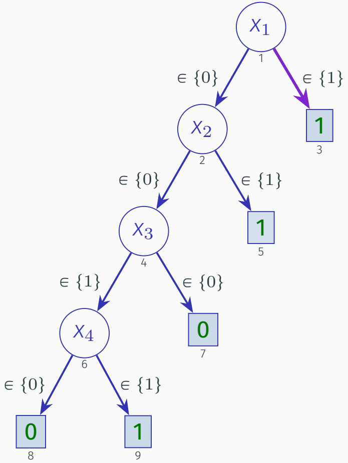

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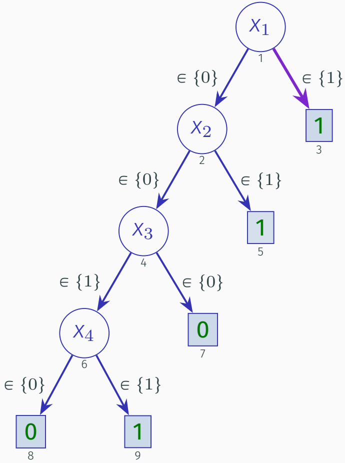
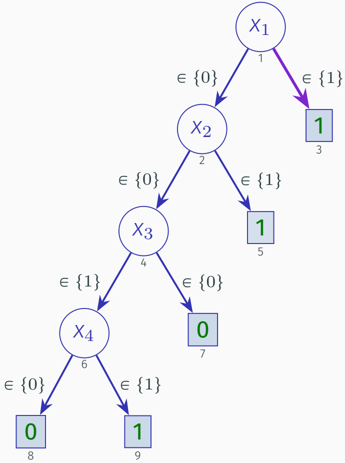
- Confirmation:
  - CXps: $\{\{1\},\{2\},\{3,4\}\}$ (2 is also AXp-necessary)
  - AXps: $\{\{1,2,3\},\{1,2,4\}\}$



Recap example – a different instance

- Instance $(\mathbf{v}, c) = ((1, 1, 1, 1), 1)$



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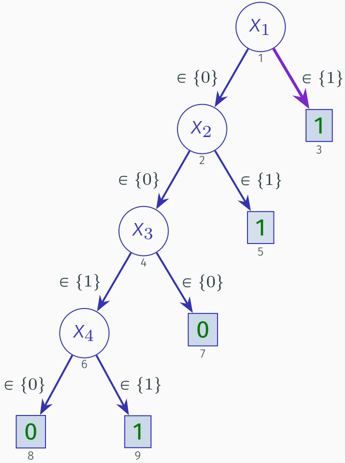
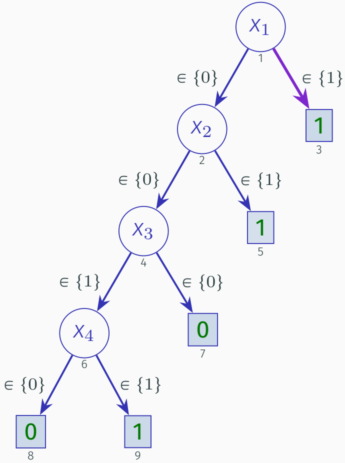


Recap example – a different instance

- Instance $(\mathbf{v}, c) = ((1,1,1,1),1)$
- Are there CXp-necessary features?
  - **Yes!** Features 1 and 2 (i.e. singleton AXps)



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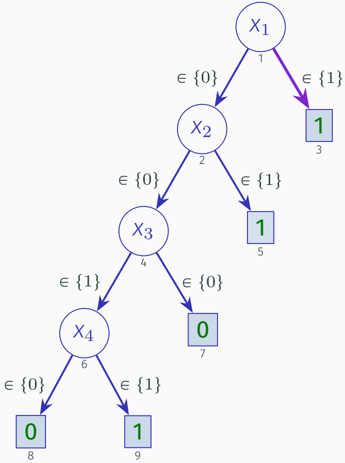

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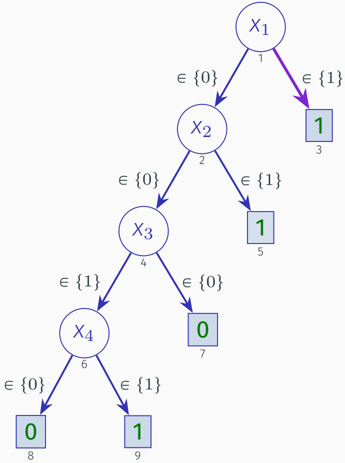

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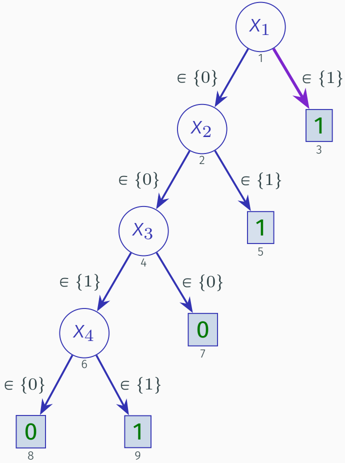

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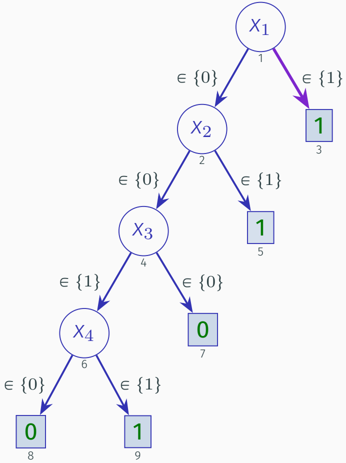
- Confirmation:
  - AXps: $\{\{1\}, \{2\}, \{3, 4\}\}$
  - CXps: $\{\{1, 2, 3\}, \{1, 2, 4\}\}$



Another example – feature necessity & relevancy

- Classifier: $\mathcal{F} = \{1,2,3,4,5\}$; $\mathcal{D}_i = \{0,1\}$, $i = 1,\ldots,5$; $\mathcal{K} = \{0,1\}$

$$\kappa(x_1,x_2,x_3,x_4,x_5) \quad := \quad \begin{cases} 1 & \text{IF } (10x_1 + 5x_2 + 5x_3 + 2x_4 + x_5 \geqslant 15) \\ 0 & \text{otherwise} \end{cases}$$

- Instance: $((1,1,1,1,1),1)$

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- Relevant:

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- Irrelevant: $\{4, 5\}$

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Yet another example – feature necessity & relevancy

- Classifier: $\mathcal{F} = \{1,2,3,4,5\}$; $\mathcal{D}_i = \{0,1\}$, $i = 1,\ldots,5$; $\mathcal{K} = \{0,1\}$

$$\kappa(x_1,x_2,x_3,x_4,x_5) \quad := \quad \begin{cases} 1 & \text{IF } (10x_1 + 5x_2 + 5x_3 + 2x_4 + x_5 \geqslant 10) \\ 0 & \text{otherwise} \end{cases}$$

- Instance: $((1,1,1,1,1),1)$
- **Obs:** If $x_1 = 0$ and $x_2 = x_3 = 0$, then $\kappa(\mathbf{x}) = 0$; i.e. must either set $x_1 = 1$ or $x_2 = x_3 = 1$
  - **Hint:** Can construct restricted truth-tables
- All AXps: $\{\{1\},\{2,3\}\}$
- All CXps: $\{\{1,2\},\{1,3\}\}$
- AXp-necessary: $\emptyset$
- CXp-necessary: $\{1\}$ (singleton AXp)
- Relevant: $\{1,2,3\}$
- Irrelevant: $\{4,5\}$

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Q: What can we do if human decision maker finds computed AXp/CXp to be unsatisfactory?

- Partially enumerate AXps/CXps, exploiting bias in enumeration

Plan for this course

- Lecture 01 – units:
  - #01: Foundations
- Lecture 02 – units:
  - #02: Principles of symbolic XAI – feature selection
  - #03: Tractability in symbolic XAI (& myth of interpretability)
- Lecture 03 – units:
  - #04: Intractability in symbolic XAI (& myth of model-agnostic XAI)
  - #05: Explainability queries
- Lecture 04 – units:
  - #06: Advanced topics
- Lecture 05 – units:
  - #07: Principles of symbolic XAI – feature attribution (& myth of Shapley values in XAI)
  - #08: Conclusions & research directions

Detour: Monotonic Classification & Voting Power

Monotonically increasing boolean classifiers

Monotonically increasing boolean classifiers

- Monotonic classifier $\mathcal{M} = (\mathcal{F}, \mathbb{F}, \mathcal{K}, \kappa)$, such that each $\mathcal{D}_i = \{0, 1\}$ and $\mathcal{K} = \{0, 1\}$ are ordered (i.e. $0 < 1$), and
  - $\kappa(\mathbf{1}) = 1$;
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  - $\kappa(\mathbf{x}_1) \leqslant \kappa(\mathbf{x}_2)$ when $\mathbf{x}_1 \leqslant \mathbf{x}_2$

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- Let $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{F}$ be such that $\kappa(\mathbf{v}_1) = \kappa(\mathbf{v}_2) = 1$, and $\mathbf{v}_1 \leqslant \mathbf{v}_2$
  Define the explanation problems:
  - $\mathcal{E}_1 = (\mathcal{M}, (\mathbf{v}_1, 1))$
  - $\mathcal{E}_2 = (\mathcal{M}, (\mathbf{v}_2, 1))$
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- Then,
  - If WAXp$(\mathcal{S}; \mathcal{E}_1)$ holds, then WAXp$(\mathcal{S}; \mathcal{E}_2)$ holds; in particular:
  - $\mathbb{A}(\mathcal{E}_{\mathbb{1}})$ contains **all** the AXps of **any** instance of the form $(\mathbf{v}_r, 1)$
    - Why?
      - Pick any explanation problem $\mathcal{E}_r$ with instance $(\mathbf{v}_r, 1)$
      - Start from $\mathbb{1} = (1, 1, \ldots, 1)$
      - Remove features that take value 0 in $\mathbf{v}_r$; we still have an WAXp
      - Then compute any AXp starting from features taking value 1 in $\mathbf{v}_r$

      $\therefore$ **Suffices to find explanations for $\mathcal{E}_{\mathbb{1}}$** (or alternatively, the global explanations for prediction 1)

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    - Feature 6: can be dropped
    - AXp: $\{2, 3, 4, 5\}$; **Q**: Is feature 6 relevant?

All AXps & all CXps...

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  - Problem: find a measure of importance of each voter !
    - I.e. measure the a priori voting power of each voter

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Coutry	Acronym	# Votes
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- Perhaps surprisingly, answer is **No**!
  - In 1958, Luxembourg was a **dummy** voter/player

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Quota:	12	

- The corresponding classifier is:

$$\kappa(x_1, x_2, x_3, x_4, x_5, x_6) \quad := \quad \begin{cases} 1 & \text{IF } (4x_1 + 4x_2 + 4x_3 + 2x_4 + 2x_5 + x_6 \geqslant 12) \\ 0 & \text{otherwise} \end{cases}$$

which we have seen before! E.g. $\{2, 3, 4, 5\}$ is an AXp & feature 6 (L) is irrelevant

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- In turn, this revealed novel connections between logic-based XAI and a priori voting power [LHAMS24]
- Homework:
  - Create your own weighted voting games;
  - Compute the sets of AXps and CXps; and
  - Assess the importance of features and how they compare to each other

Unit #06

Advanced Topics

Outline – Unit #06

Changing Assumptions

Inflated Explanations

Probabilistic Explanations

Constrained Explanations

Distance-Restricted Explanations

Additional Topics

General definition of prediction sufficiency

- Instance $(\mathbf{v}, c)$
- Let $\mathcal{S} \subseteq \mathcal{F}$:
  - Recall,

$$\Upsilon(\mathcal{S};\mathbf{v}) = \{\mathbf{x} \in \mathbb{F} \mid \mathbf{x}_{\mathcal{S}} = \mathbf{v}_{\mathcal{S}}\}$$

- $\mathcal{S} \subseteq \mathcal{F}$ suffices for prediction c if:

$$\forall(\mathbf{x} \in \mathbb{F}).(\mathbf{x} \in \Upsilon(\mathcal{S};\mathbf{v})) \rightarrow (\sigma(\mathbf{x}))$$

  - **Obs:** a WAXp is just one possible example
- But there are other ways to study prediction sufficiency:
  - One can envision defining other sets of points Γ, parameterized by $\mathcal{E} = (\mathcal{M}, (\mathbf{v}, c))$; $\mathcal{S} \subseteq \mathcal{F}$ suffices for prediction c if:

$$\forall(\mathbf{x} \in \mathbb{F}).(\mathbf{x} \in \Gamma(\mathcal{S};\mathcal{E})) \rightarrow (\sigma(\mathbf{x}))$$

    - And one can also envision generalizations of σ!

Outline – Unit #06

Towards more expressive explanations – inflated explanations

[IISM24]

- Recall:

$$\mathsf{WAXp}(\mathcal{X}) \quad := \quad \forall(\mathbf{x} \in \mathbb{F}). \bigwedge\nolimits_{j \in \mathcal{X}} (x_j = v_j) \rightarrow (\kappa(\mathbf{x}) = c)$$

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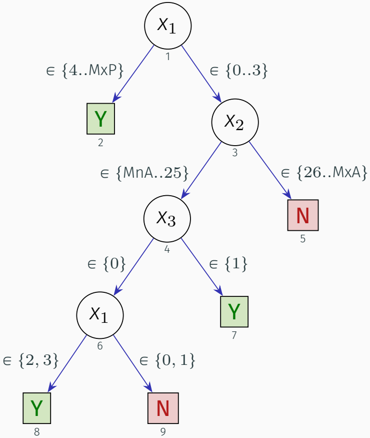
- Inflated explanations allow for more expressive literals, i.e. $=$ replaced with $\in$, and individual values replaced by ranges of values
  - Operational definition: Given an AXp, expand set of values of each feature, in some chosen order, such that the set of picked features remains unchanged

Inflated explanations – an example

[IIM22]



- Explanation for $((2, 20, 0), \mathsf{Y})$? (Obs: $\mathsf{MnA} = 18; \mathsf{MxP} > 4$)

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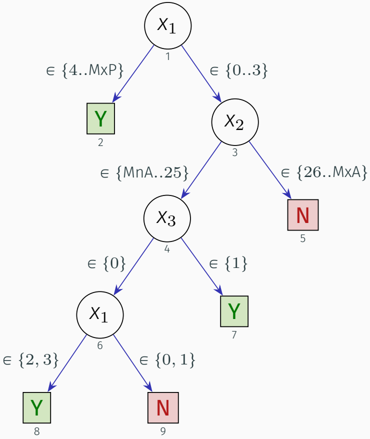

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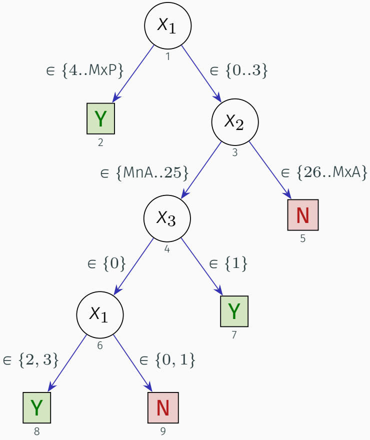

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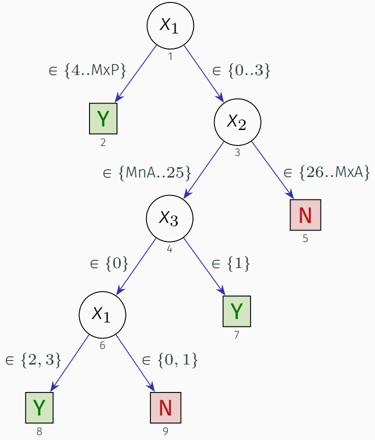

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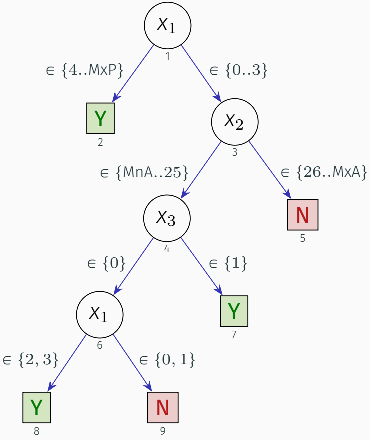

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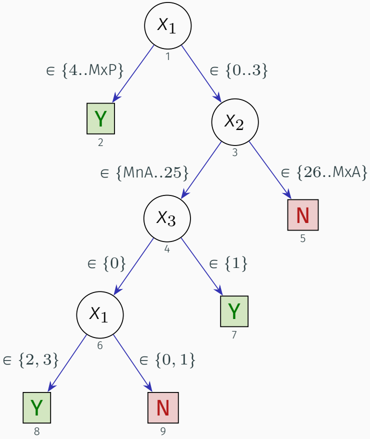

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Approach

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- For each feature:
  - Categorical: iteratively add elements to literal
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- **Obs:** More complex alternative is to find AXp and expand domains simultaneously
  - This is conjectured to change the complexity class of finding one explanation

Outline – Unit #06

Probabilistic (formal) explanations

[WMHK21, IIN+22, IHI+22, ABOS22, IHI+23, IMM24]

- Explanation size is critical for human understanding [Mil56]

- Probabilistic explanations provide smaller explanations, by trading off rigor of explanation by explanation size

Probabilistic (formal) explanations

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- Explanation size is critical for human understanding [Mil56]
- Probabilistic explanations provide smaller explanations, by trading off rigor of explanation by explanation size
- Definition of weak probabilistic AXp $\mathcal{X} \subseteq \mathcal{F}$:

$$\mathsf{WPAXp}(\mathcal{X}) \quad := \quad \mathbf{Pr}(\kappa(\mathbf{x}) = c) \,|\, \mathbf{x}_{\mathcal{X}} = \mathbf{v}_{\mathcal{X}}) \geqslant \delta$$

  - Obs: $\mathbf{x}_{\mathcal{X}} = \mathbf{v}_{\mathcal{X}}$ requires points $\mathbf{x} \in \mathbb{F}$ to match the values of $\mathbf{v}$ for the features dictated by $\mathcal{X}$
  - Obs: for $\delta = 1$ we obtain a WAXp

Definitions

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$$\mathsf{WeakPAXp}(\mathcal{X};\mathbb{F},\kappa,\mathbf{v},c,\delta) :=$$

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Results for decision trees

Dataset	DT N	DT A	Path M	Path m	Path avg	δ	MinPAXp Length M	MinPAXp Length m	MinPAXp Length avg	MinPAXp Prec avg	MinPAXp Time avg	LmPAXp Length M	LmPAXp Length m	LmPAXp Length avg	LmPAXp Prec avg	LmPAXp $m_{\subseteq}$	LmPAXp Time avg	Anchor D	Anchor Length M	Anchor Length m	Anchor Length avg	Anchor Length $F_{\notin P}$	Anchor Prec avg	Anchor Time avg
						100	11	3	6.8	100	2.34	11	3	6.9	100	100	0.00	d	12	2	7.0	26.8	76.8	0.96
adult	1241	89	14	3	10.7	95	11	3	6.2	98.4	5.36	11	3	6.3	98.6	99.0	0.01	u	12	3	10.0	29.4	93.7	2.20
						90	11	2	5.6	94.6	4.64	11	2	5.8	95.2	96.4	0.01							
						100	12	1	4.4	100	0.35	12	1	4.4	100	100	0.00	d	31	1	4.8	58.1	32.9	3.10
dermatology	71	100	13	1	5.1	95	12	1	4.1	99.7	0.37	12	1	4.1	99.7	99.3	0.00	u	34	1	13.1	43.2	87.2	25.13
						90	11	1	4.0	98.8	0.35	11	1	4.0	98.8	100	0.00							
						100	12	2	4.8	100	0.93	12	2	4.9	100	100	0.00	d	36	2	7.9	44.8	69.4	1.94
kr-vs-kp	231	100	14	3	6.6	95	11	2	3.9	98.1	0.97	11	2	4.0	98.1	100	0.00	u	12	2	3.6	16.6	97.3	1.81
						90	10	2	3.2	95.4	0.92	10	2	3.3	95.4	99.0	0.00							
						100	12	4	8.2	100	16.06	11	4	8.2	100	100	0.00	d	16	3	13.2	43.1	71.3	12.22
letter	3261	93	14	4	11.8	95	12	4	8.0	99.6	18.28	11	4	8.0	99.5	100	0.00	u	16	3	13.7	47.3	66.3	10.15
						90	12	4	7.7	97.7	16.35	10	4	7.8	97.8	100	0.00							
						100	14	3	6.4	100	0.92	14	3	6.5	100	100	0.00	d	35	2	8.6	55.4	33.6	5.43
soybean	219	100	16	3	7.3	95	14	3	6.4	99.8	0.95	14	3	6.4	99.8	100	0.00	u	35	3	19.2	66.0	75.0	38.96
						90	14	3	6.1	98.1	0.94	14	3	6.1	98.2	98.5	0.00							
						0	12	3	7.4	100	1.23	12	3	7.5	100	100	0.01	d	38	2	6.3	65.3	63.3	24.12
spambase	141	99	14	3	8.5	95	9	1	3.7	96.1	2.16	9	1	3.8	96.5	100	0.01	u	57	3	28.0	86.2	65.3	834.70
						90	6	1	2.4	92.4	2.15	8	1	2.4	92.2	100	0.01							

Results for naive Bayes classifiers

Dataset	(#F #I)	NBC	AXp		$\text{LmPAXp}_{\leqslant 9}$				$\text{LmPAXp}_{\leqslant 7}$				$\text{LmPAXp}_{\leqslant 4}$			
		A%	Length	δ	Length	Precision	W%	Time	Length	Precision	W%	Time	Length	Precision	W%	Time
adult	(13 200)	81.37	6.8± 1.2	98	6.8± 1.1	100± 0.0	100	0.003	6.3± 0.9	99.61± 0.6	96	0.023	4.8± 1.3	98.73± 0.5	48	0.059
				95	6.8± 1.1	99.99± 0.2	100	0.074	5.9± 1.0	98.87± 1.8	99	0.058	3.9± 1.0	96.93± 1.1	80	0.071
				93	6.8± 1.1	99.97± 0.4	100	0.104	5.7± 1.3	98.34± 2.6	100	0.086	3.4± 0.9	95.21± 1.6	90	0.093
				90	6.8± 1.1	99.95± 0.6	100	0.164	5.5± 1.4	97.86± 3.4	100	0.100	3.0± 0.8	93.46± 1.5	94	0.103
agaricus	(23 200)	95.41	10.3± 2.5	98	7.7± 2.7	99.12± 0.8	92	0.593	6.4± 3.0	98.75± 0.6	87	0.763	6.0± 3.1	98.67± 0.5	29	0.870
				95	6.9± 3.1	97.62± 2.1	95	0.954	5.3± 3.2	96.59± 1.6	92	1.273	4.8± 3.3	96.24± 1.2	55	1.217
				93	6.5± 3.1	96.65± 2.8	95	1.112	4.8± 3.1	95.38± 1.9	93	1.309	4.3± 3.1	94.92± 1.3	64	1.390
				90	5.9± 3.3	94.95± 4.1	96	1.332	4.0± 3.0	92.60± 2.8	95	1.598	3.6± 2.8	92.08± 1.7	76	1.830
chess	(37 200)	88.34	12.1± 3.7	98	8.1± 4.1	99.27± 0.6	64	0.383	5.9± 4.9	98.70± 0.4	64	0.454	5.7± 5.0	98.65± 0.4	46	0.457
				95	7.7± 3.8	98.51± 1.4	68	0.404	5.5± 4.4	97.90± 0.9	64	0.483	5.3± 4.5	97.85± 0.8	46	0.478
				93	7.3± 3.5	97.56± 2.4	68	0.419	5.0± 4.1	96.26± 2.2	64	0.485	4.8± 4.1	96.21± 2.1	64	0.493
				90	7.3± 3.5	97.29± 2.9	70	0.413	4.9± 4.0	95.99± 2.6	64	0.483	4.8± 4.0	95.93± 2.5	64	0.543
vote	(17 81)	89.66	5.3± 1.4	98	5.3± 1.4	100± 0.0	100	0.000	5.3± 1.3	99.95± 0.2	100	0.007	4.6± 1.1	99.60± 0.4	64	0.014
				95	5.3± 1.4	100± 0.0	100	0.000	5.3± 1.3	99.93± 0.3	100	0.008	4.1± 1.0	98.25± 1.7	64	0.018
				93	5.3± 1.4	100± 0.0	100	0.000	5.2± 1.3	99.78± 1.1	100	0.012	4.1± 0.9	98.10± 1.9	64	0.018
				90	5.3± 1.4	100± 0.0	100	0.000	5.2± 1.3	99.78± 1.1	100	0.012	4.0± 1.2	97.24± 3.1	64	0.022
kr-vs-kp	(37 200)	88.07	12.2± 3.9	98	7.8± 4.2	99.19± 0.5	64	0.387	6.5± 4.7	98.99± 0.4	64	0.427	6.1± 4.9	98.88± 0.3	43	0.457
				95	7.3± 3.9	98.29± 1.4	64	0.416	6.0± 4.3	97.89± 1.1	64	0.453	5.5± 4.5	97.79± 0.9	43	0.462
				93	6.9± 3.5	97.21± 2.5	69	0.422	5.6± 3.8	96.82± 2.2	64	0.448	5.2± 4.0	96.71± 2.1	43	0.468
				90	6.8± 3.5	96.65± 3.1	69	0.418	5.4± 3.8	95.69± 3.0	64	0.468	5.0± 4.0	95.59± 2.8	61	0.487
mushroom	(23 200)	95.51	10.7± 2.3	98	7.5± 2.4	98.99± 0.7	90	0.641	6.5± 2.6	98.74± 0.5	83	0.751	6.3± 2.7	98.70± 0.4	18	0.828
				95	6.5± 2.6	97.35± 1.8	96	1.011	5.1± 2.5	96.52± 1.0	90	1.130	5.0± 2.5	96.39± 0.8	54	1.113
				93	5.8± 2.8	95.77± 2.7	96	1.257	4.4± 2.5	94.67± 1.6	94	1.297	4.2± 2.4	94.48± 1.3	65	1.324

Results for decision diagrams

Dataset	#I	#F	OMDD		δ	MinPAXp					LmPAXp					
						Length			Prec	Time	Length			Prec	$m_{\subseteq}$	Time
			#N	A%		M	m	avg	avg	avg	M	m	avg	avg		avg
lending	100	9	1103	81.7	100	9	6	8.0	100	24.24	9	6	7.9	100	100	1.57
					95	9	5	7.8	99.7	21.48	9	6	7.8	99.8	100	1.49
					90	9	4	7.2	96	24.65	9	5	7.4	97.0	100	1.48
monk2	100	6	70	79.3	100	6	4	5.1	100	0.10	6	4	5.1	100	100	0.03
					95	6	4	5.1	100	0.09	6	4	5.1	100	100	0.03
					90	6	3	4.8	98.1	0.09	6	3	4.8	98.1	100	0.03
postoperative	74	8	109	80	100	8	4	6.1	100	0.26	8	4	6.2	100	100	0.04
					95	8	2	6.0	99.3	0.25	8	2	6.0	99.3	100	0.04
					90	8	2	5.3	95.9	0.23	8	2	5.4	96.6	94.6	0.04
tic_tac_toe	100	9	424	70.3	100	9	5	7.7	100	3.60	9	5	7.8	100	100	0.38
					95	9	5	7.5	99.5	3.24	9	5	7.7	99.6	99.0	0.38
					90	9	3	7.3	98.3	4.06	9	3	7.5	98.6	98.0	0.38
xd6	100	9	76	83.1	100	9	4	4.6	100	0.10	9	4	4.6	100	100	0.03
					95	9	3	3.8	97	0.09	9	3	3.8	97.0	99.0	0.03
					90	9	3	3.3	94.8	0.10	9	3	3.4	94.6	100	0.03

Remarks on LmPAXps

[IHI+23]

- LmPAXps ignore non-monotonicity, and so overapproximate PAXps
  - Theoretical guarantees, but may be reducible
- For DTs, computation of LmPAXps is in P
- Experimental results confirm LmPAXps match PAXps in most cases
- Recent results on approximating LmPAXps for RFs [IMM24]

Outline – Unit #06

Not all inputs may be possible – input constraints

[GR22, YIS+23]

- The (implicit) assumption that all inputs are possible is often unrealistic
  - I.e. it may be impossible for some points in feature space to be observed

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$$\exists(\mathbf{x} \in \mathbb{F}).\left[\bigwedge_{j\in\mathcal{X}}(x_j = v_j) \land \mathcal{C}(\mathbf{x})\right] \land (\kappa(\mathbf{x}) \neq c)$$

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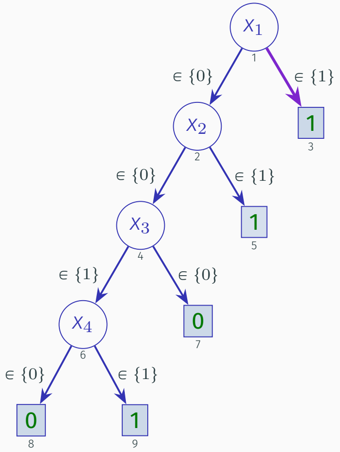
  - Compute AXps/CXps given new definitions
- Constrained AXps/CXps find other applications!

An example

- Instance: $((1,1,1,1),1)$
- Unconstrained AXps:



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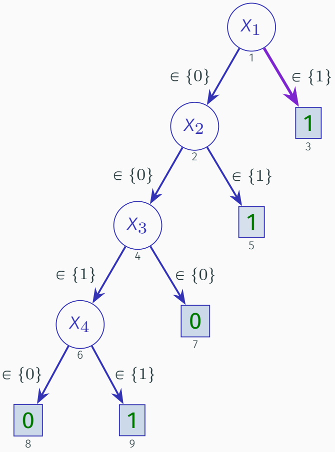

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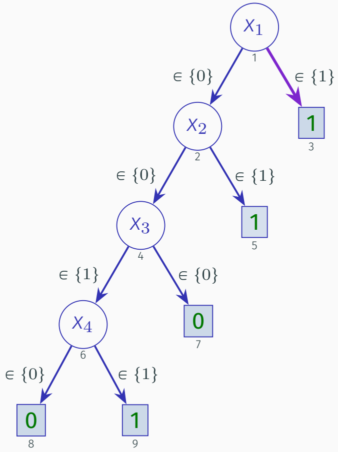

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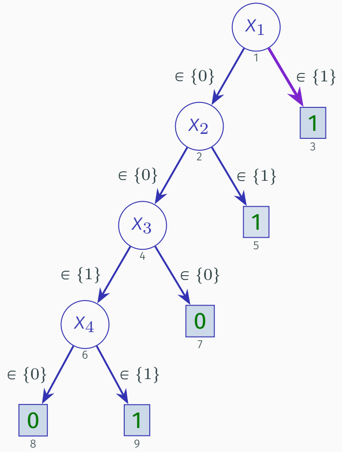

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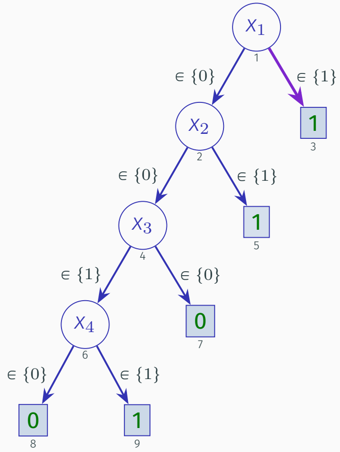

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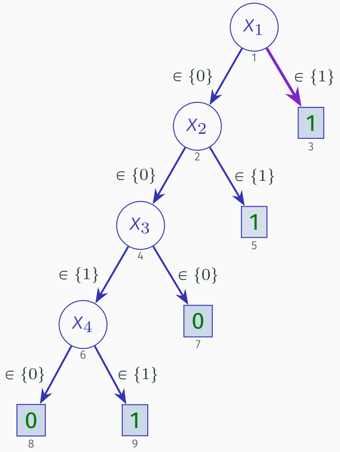

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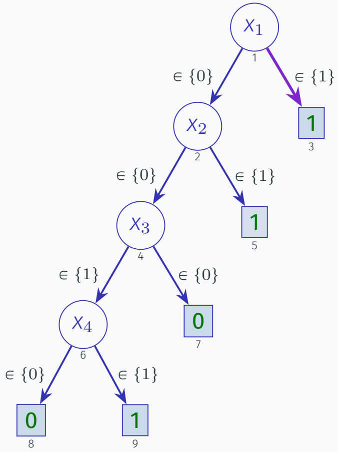

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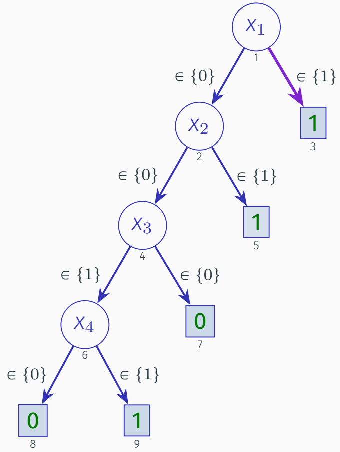

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Outline – Unit #06

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- Change definition of WAXp/WCXp to account for l_p distance to $\mathbf{v}$:

$$\forall(\mathbf{x} \in \mathbb{F}).\left[\bigwedge\nolimits_{j\in\mathcal{X}}(x_j = v_j) \land \left(\|\mathbf{x}-\mathbf{v}\|_{l_p} \leqslant \epsilon\right)\right] \rightarrow (\sigma(\mathbf{x}))$$

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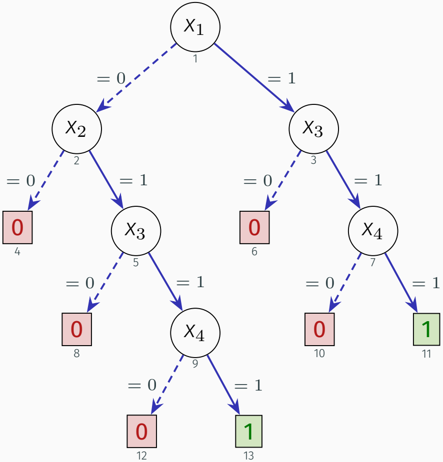
  - Norm l_p is arbitrary, e.g. Hamming, Manhattan, Euclidean, etc.
  - **Distance-restricted explanations**: ∂AXp/∂CXp

An example – DT & instance $((1,1,1,1),1)$



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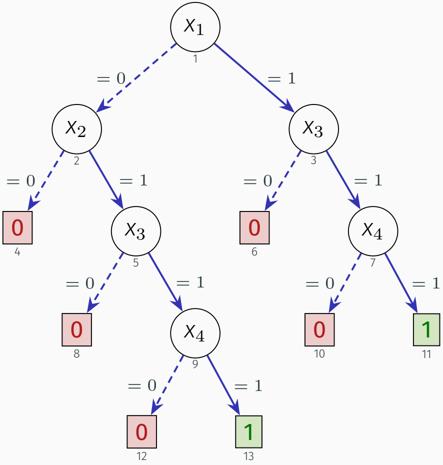

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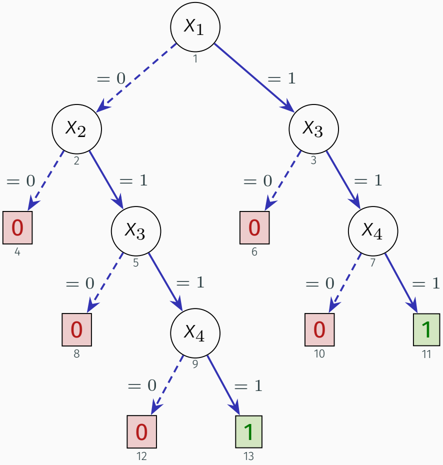

- Plain AXps/CXps:
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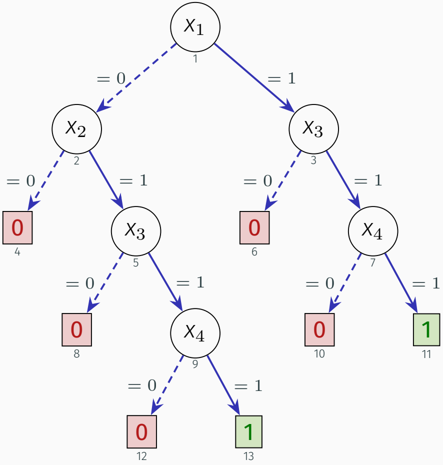

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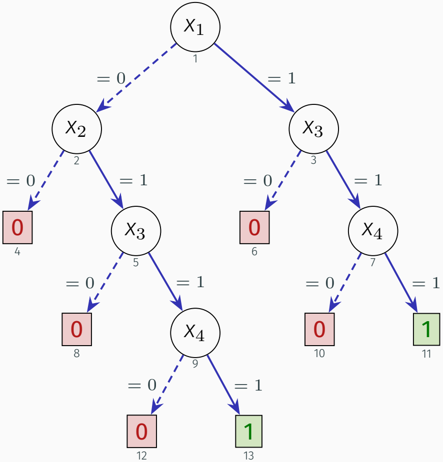

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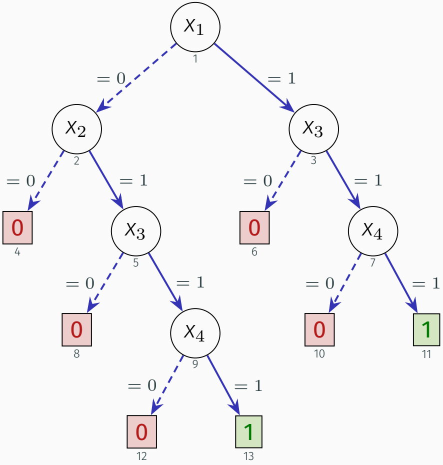

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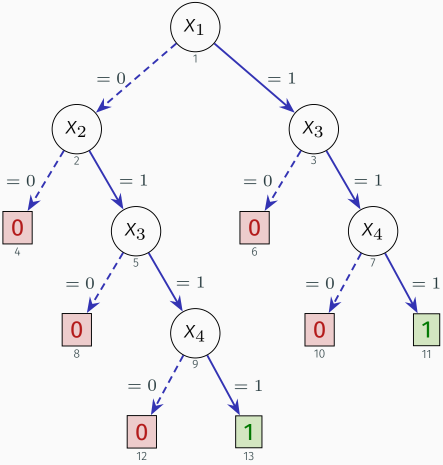

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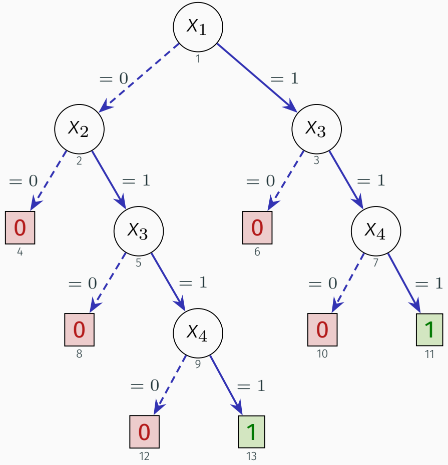

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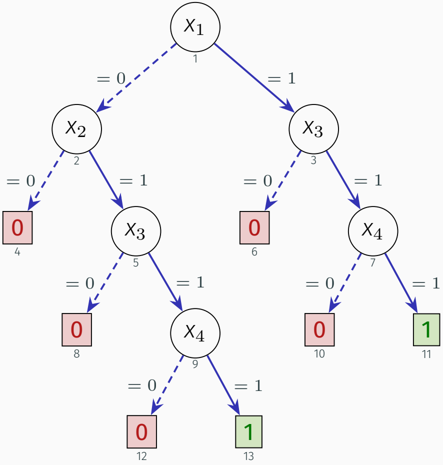

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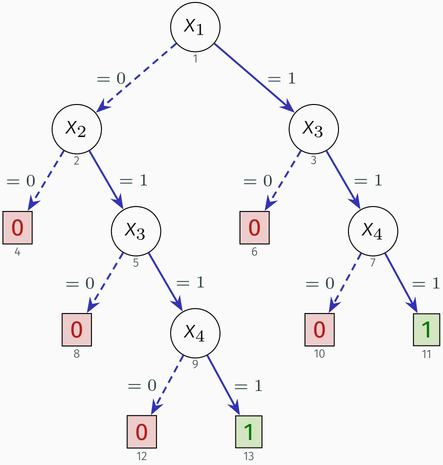

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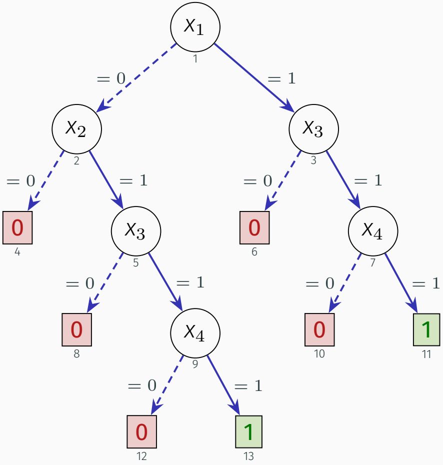
- Given ϵ, larger adversarial examples are excluded



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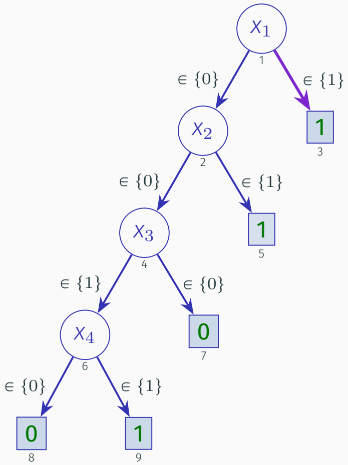

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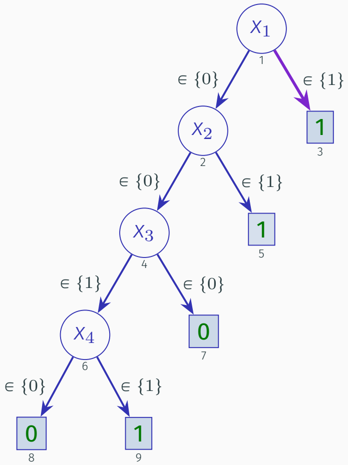

Another example – DT & instance $((1,1,1,1),1)$

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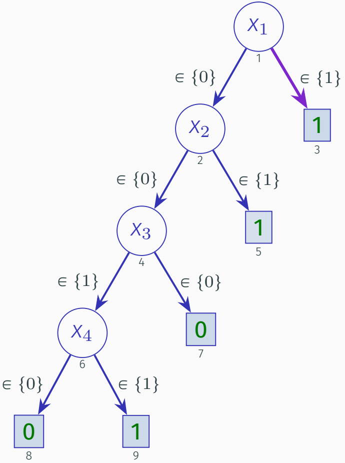

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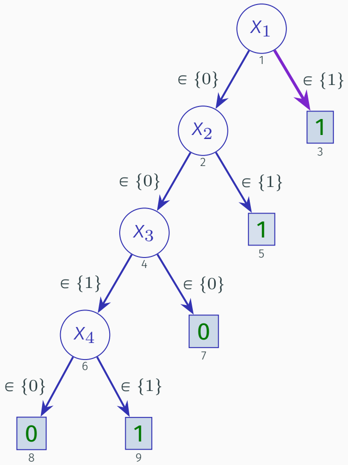

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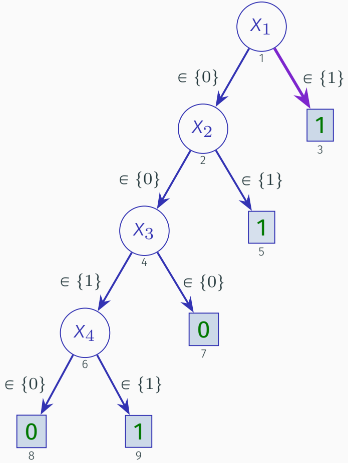

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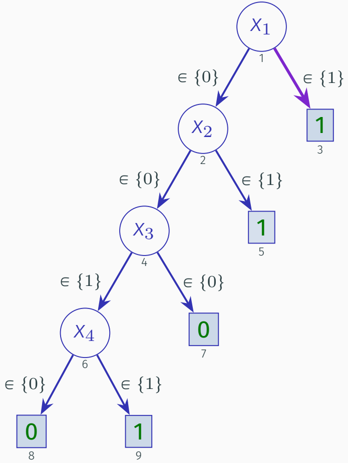

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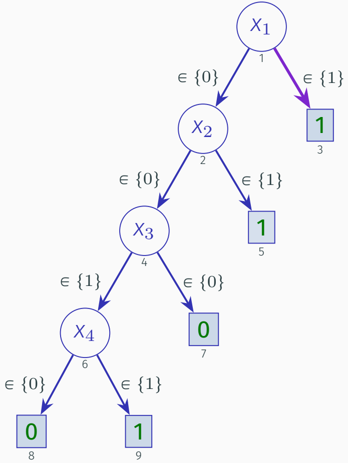

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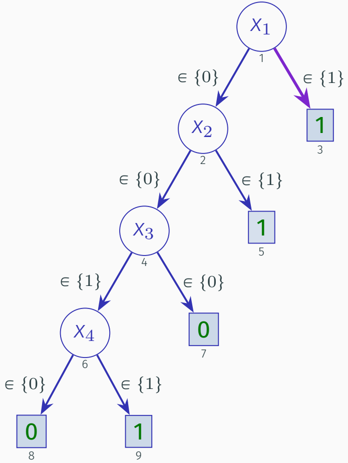

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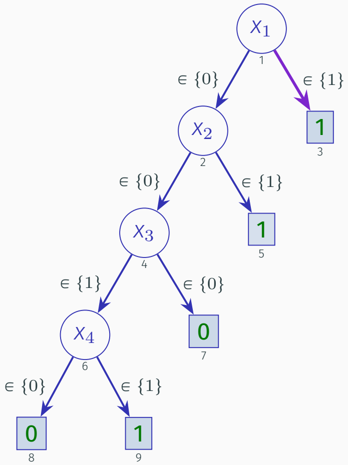

- Distance-restricted WAXps/WCXps:

$$\forall(\mathbf{x}\in\mathbb{F}).\left[\bigwedge\nolimits_{j\in\mathcal{X}}(x_j=v_j)\land\left(\|\mathbf{x}-\mathbf{v}\|_{l_p}\leqslant\epsilon\right)\right]\rightarrow(\sigma(\mathbf{x}))$$

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Relating explanations with adversarial examples

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- Clear scalability improvements for explaining NNs (see next) [HM23b, WWB23, IHM+24a, IHM+24b]

Basic algorithm

[HM23b]

Input: Arguments: ϵ; Parameters: $\mathcal{E}$, p

Output: One ∂AXp $\mathcal{S}$

function FindAXpDel($\epsilon; \mathcal{E}, p$)
  $\mathcal{S} \leftarrow \mathcal{F}$ ▷ Initially, no feature is allowed to change
  for $i \in \mathcal{F}$ **do** ▷ Invariant: ∂WAXp($\mathcal{S}$)
    $\mathcal{S} \leftarrow \mathcal{S} \setminus \{i\}$
    outc $\leftarrow$ FindAdvEx($\epsilon, \mathcal{S}; \mathcal{E}, p$)
    if outc **then**
      $\mathcal{S} \leftarrow \mathcal{S} \cup \{i\}$
  return $\mathcal{S}$ ▷ ∂WAXp($\mathcal{S}$) $\wedge$ minimal($\mathcal{S}$) $\rightarrow$ ∂AXp($\mathcal{S}$)

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- **Obs:** Efficiency of logic-based XAI tracks efficiency of robustness tools

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[HM23b]

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function FindAXpDel(ϵ; ℰ, p)
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```

- **Obs:** Efficiency of logic-based XAI tracks efficiency of robustness tools
- **Limitation:** Running time grows with number of features

Results for NNs in 2023 (using Marabou [KHI+19])

DNN	points	|AXp|	#Calls	Time	#TO	|AXp|	#Calls	Time	#TO
			$\epsilon = 0.1$				$\epsilon = 0.05$		
ACASXU_1_5	#1	3	5	185.9	0	2	5	113.8	0
	#2	2	5	273.8	0	1	5	33.2	0
	#3	0	5	714.2	0	0	5	4.3	0
ACASXU_3_1	#1	0	5	2219.3	0	0	5	14.2	0
	#2	2	5	4263.5	1	0	5	1853.1	0
	#3	1	5	581.8	0	0	5	355.9	0
ACASXU_3_2	#1	3	5	13739.3	2	1	5	6890.1	1
	#2	3	5	226.4	0	2	5	125.1	0
	#3	2	5	1740.6	0	2	5	173.6	0
ACASXU_3_5	#1	4	5	43.6	0	2	5	59.4	0
	#2	3	5	5039.4	0	2	5	4303.8	1
	#3	2	5	5574.9	1	2	5	2660.3	0
ACASXU_3_6	#1	1	5	6225.0	1	0	5	51.0	0
	#2	3	5	4957.2	1	2	5	1897.3	0
	#3	1	5	196.1	0	1	5	919.2	0
ACASXU_3_7	#1	3	5	6256.2	0	4	5	26.9	0
	#2	4	5	311.3	0	1	5	6958.6	1
	#3	2	5	7756.5	1	1	5	7807.6	1
ACASXU_4_1	#1	2	5	12413.0	2	1	5	5090.5	1
	#2	1	5	5035.1	1	0	5	2335.6	0
	#3	4	5	1237.3	0	4	5	1143.4	0
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Scales to a few **hundred** neurons

Recent improvements

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    - Algorithm FindAXpDel is mostly sequential (see above)
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  - Algorithm FindAXpDel is mostly sequential (see above)
  - Exploit parallelization for other algorithms, e.g. dichotomic search [IHM+24b]
- However, to decide whether $\mathcal{S}$ is an AXp, we can exploit parallelization:
  - Recall: $\text{AXp}(\mathcal{X}) := \text{WAXp}(\mathcal{X}) \land \forall(t \in \mathcal{X}).\neg\text{WAXp}(\mathcal{X}\setminus\{t\})$
  - Each $\neg$WAXp($\cdot$) (and also WAXp($\cdot$)) check can be run in parallel!
  - Do this opportunistically, i.e. when set $\mathcal{S}$ is expected to be AXp [IHM+24b]

More recent results (from 2024)...

[IHM+24a, IHM+24b]

Model	Deletion							SwiftXplain						
	avgC	nCalls	Len	Mn	Mx	avg	TO	avgC	nCalls	Len	FD%	Mn	Mx	avg
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gtsrb-convSmall	0.06	1024	309	59.2	82.6	65.1	0	0.22	74	313	39.7	15.1	19.5	16.2
gtsrb-conv	—	—	—	—	—	—	100	96.49	45	174	33.2	3858.7	6427.7	4449.4
mnist-denseSmall	0.28	784	177	190.9	420.3	220.4	0	0.77	111	180	15.5	77.6	104.4	85.1
mnist-dense	0.19	784	231	138.1	179.9	150.6	0	0.75	183	229	11.5	130.1	145.5	136.8
mnist-convSmall	—	—	—	—	—	—	100	98.56	52	116	21.3	4115.2	6858.3	5132.8

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Largest for MNIST: **10142** neurons

Largest for GSTRB: **94308** neurons

Outline – Unit #06

Surrogate models in logic-based XAI

[BAMT21]

- Motivation:
  - Logic-based XAI does not yet scale for highly complex ML models
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Certified explainer (for monotonic classification)

[HM23f]

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- Certification envisioned for **any** explainability algorithm

Plan for this course – light at the end of the tunnel...

- Lecture 01 – units:
  - #01: Foundations
- Lecture 02 – units:
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  - #03: Tractability in symbolic XAI (& myth of interpretability)
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  - #07: Principles of symbolic XAI – feature attribution (& myth of Shapley values in XAI)
  - #08: Conclusions & research directions

Questions?

Lecture 05

Recapitulate fourth lecture

- Monotonic classifiers vs. weighted voting games
- Advanced topics:
  - Inflated explanations
  - Probabilistic explanations
  - Constrained explanations
  - Distance-restricted explanations
  - Explanations using surrogate models
  - Certified explainability

Monotonicity & WCGs

- Every WVG $\mathcal{G}$, described by $[q; n_1, \ldots, n_m]$, can be represented as a monotonically increasing boolean classifier $\mathcal{M} = (\mathcal{F}, \{0,1\}^m, \{0,1\}, \kappa)$, such that:
  - Each voter i is mapped to a boolean feature i, such that feature i takes value 1 if voter i votes Yes; otherwise it takes value 0;
  - The classification function $\kappa : \mathbb{F} \to \{0,1\}$ is defined by:

$$\kappa(\mathbf{x}) = \begin{cases} 1 & \text{if } \sum_{i=1}^{m} n_i x_i \geqslant q \\ 0 & \text{otherwise} \end{cases}$$

  - The target instance is $(\mathbb{1}, 1)$; and
  - Each minimal winning coalition $\mathcal{C}$ corresponds to an AXp of $\mathcal{E} = (\mathcal{M}, (\mathbb{1}, 1))$

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$\therefore$ WVGs can be analyzed by studying the AXps/CXps of monotonically increasing boolean classifiers

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- **Q:** How should features be ranked in terms of importance?

Plan for this course – light at the end of the tunnel...

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Unit #07

Principles of Symbolic XAI – Feature Attribution

Outline – Unit #07

Detour: Standard SHAP Intro (from another course...)

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- Shapley values are becoming ubiquitous in XAI... – E.g. see slides from other XAI course...

https://en.wikipedia.org/wiki/Shapley_value

Accessed 2023/06/14

In machine learning [edit]

The Shapley value provides a principled way to explain the predictions of nonlinear models common in the field of machine learning. By interpreting a model trained on a set of features as a value function on a coalition of players, Shapley values provide a natural way to compute which features contribute to a prediction.[17] This unifies several other methods including Locally Interpretable Model-Agnostic Explanations (LIME),[18] DeepLIFT,[19] and Layer-Wise Relevance Propagation.[20]

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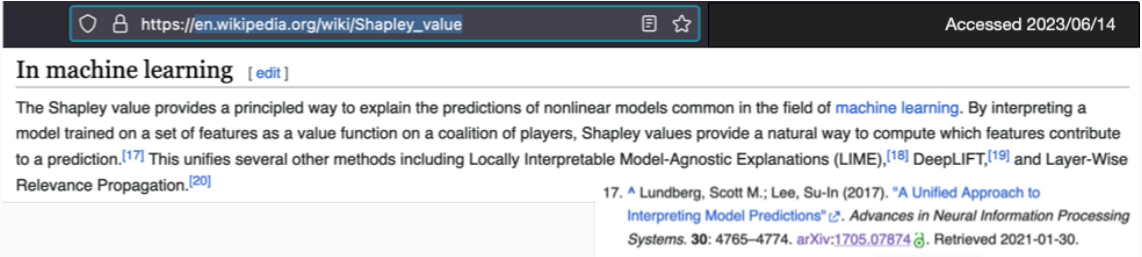
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- **Q:** Do Shapley values for XAI **really** provide a rigorous measure of feature importance?

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$$\text{Sc}(i) = \sum_{\mathcal{S} \subseteq (\mathcal{F}\setminus\{i\})} \frac{|\mathcal{S}|!(|\mathcal{F}| - |\mathcal{S}| - 1)!}{|\mathcal{F}|!} \times (\phi(\mathcal{S} \cup \{i\}) - \phi(\mathcal{S}))$$

For all subsets of features, excluding i, compute the expected value of the classifier, with and without i fixed, weighted by $\frac{1}{n}\binom{n}{|\mathcal{S}|}^{-1}$

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Marginal contribution (in SHAP lingo)!

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How are Shapley values computed in practice?

- Exact evaluation is computationally (very) hard [VLSS21, ABBM21, VLSS22, ABBM23, HMS24]
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- Polynomial-time algorithm for deterministic decomposable boolean circuits [ABBM21]
- Polynomial-time algorithm for boolean functions represented with a truth-table [HM23c]

What do Shapley values tell in terms of feature importance?

- [SK10] reads:
  "According to the 2nd axiom, if two features values have an identical influence on the prediction they are assigned contributions of equal size. The 3rd axiom says that if a ***feature has no influence*** *on the prediction* ***it is assigned a contribution of 0****."*
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  - **Qs**: can we have **irrelevant** features with a non-zero Shapley value, and/or **relevant** features with a Shapley of zero?
    - Recall: **relevant** features occur in **some** AXp/CXp; **irrelevant** features do **not** occur in **any** AXp/CXp

Outline – Unit #07

Exact Shapley Values for XAI

Myth #03: Shapley Values for XAI

Corrected SHAP Scores

Voting Power & Power Indices

Feature Importance Scores

Shapley values vs. feature (ir)relevancy – identified issues [HM23c, HM23d, HM23e, MH23, HMS24, MSH24]

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Any of these issues is a cause of (**serious**) concern per se**!**

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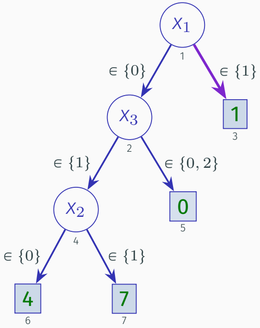
Some stats – all boolean functions with 4 variables

[HM23c, HM23d, HM23e, MH23, HMS24, MSH24]

Issue-related metric	Value	Recap issue
# of functions	65536	
# number of instances	1048576	
# of I1 issues	781696	
# of functions with I1 issues	65320	
% I1 issues / function	99.67	$[\mathsf{Irrelevant}(i) \wedge (\mathsf{Sv}(i) \neq 0)]$
# of I2 issues	105184	
# of functions with I2 issues	40448	
% I2 issues / function	61.72	$[\mathsf{Irrelevant}(i_1) \wedge \mathsf{Relevant}(i_2) \wedge (\lvert\mathsf{Sv}(i_1)\rvert > \lvert\mathsf{Sv}(i_2)\rvert)]$
# of I3 issues	43008	
# of functions with I3 issues	7800	
% I3 issues / function	11.90	$[\mathsf{Relevant}(i) \wedge (\mathsf{Sv}(i) = 0)]$
# of I4 issues	5728	
# of functions with I4 issues	2592	
% I4 issues / function	3.96	$[\mathsf{Irrelevant}(i_1) \wedge (\mathsf{Sv}(i_1) \neq 0)] \wedge [\mathsf{Relevant}(i_2) \wedge (\mathsf{Sv}(i_2) = 0)]$
# of I5 issues	1664	
# of functions with I5 issues	1248	
% I5 issues / function	1.90	$[\mathsf{Irrelevant}(i) \wedge \forall_{1 \leqslant j \leqslant m, j \neq i} (\lvert\mathsf{Sv}(j)\rvert < \lvert\mathsf{Sv}(i)\rvert)]$

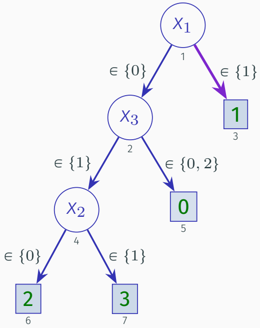
Previous results do matter! Let's go non-boolean...



DT1

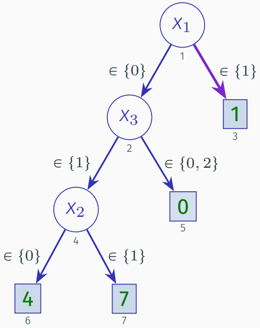
row #	x_1	x_2	x_3	$\kappa_1(\mathbf{x})$	$\kappa_2(\mathbf{x})$
1	0	0	0	0	0
2	0	0	1	4	2
3	0	0	2	0	0
4	0	1	0	0	0
5	0	1	1	7	3
6	0	1	2	0	0
7	1	0	0	1	1
8	1	0	1	1	1
9	1	0	2	1	1
10	1	1	0	1	1
11	1	1	1	1	1
12	1	1	2	1	1

Tabular representations



DT2

Instance $((1,1,2),1)$ – which feature matters the most for prediction 1?



DT1

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2	0	0	1	4	2
3	0	0	2	0	0
4	0	1	0	0	0
5	0	1	1	7	3
6	0	1	2	0	0
7	1	0	0	1	1
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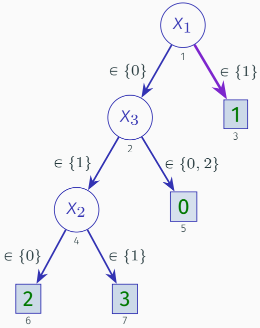
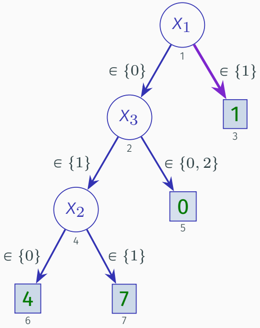


DT2

Computing XPs – make sense...



DT1

row #	x_1	x_2	x_3	$\kappa_1(\mathbf{x})$	$\kappa_2(\mathbf{x})$
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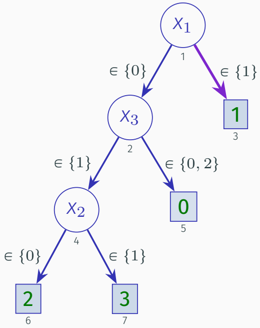
Tabular representations



DT2

XPs: AXps/CXps		
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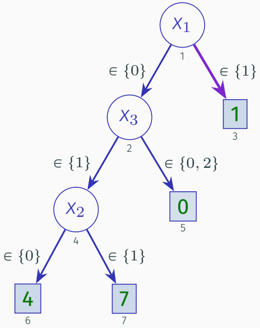
Computing XPs, AEs – also make sense...



DT1

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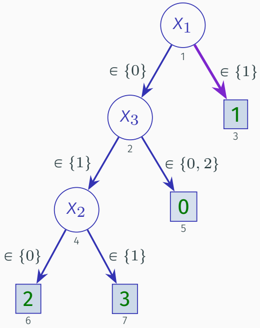

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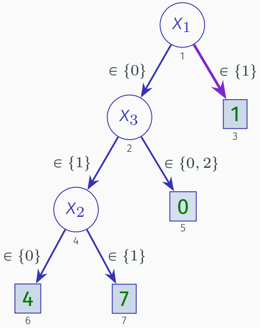

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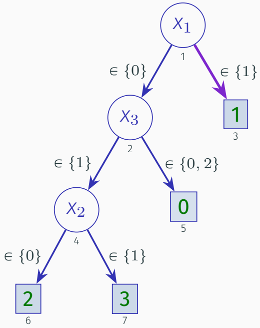

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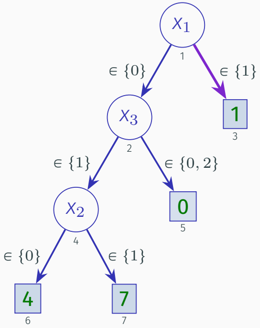
Computing XPs, AEs & Svs – what???



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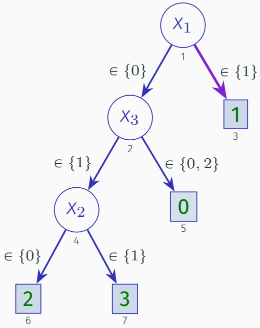

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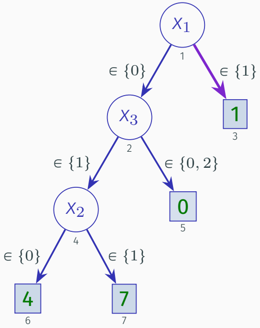

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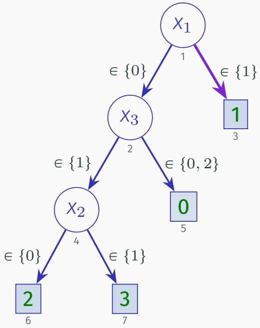

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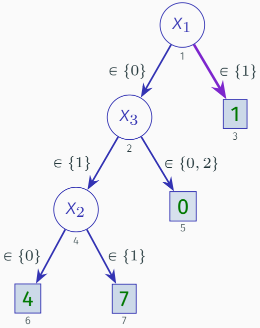

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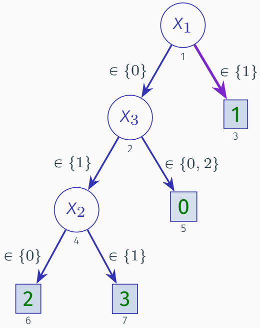

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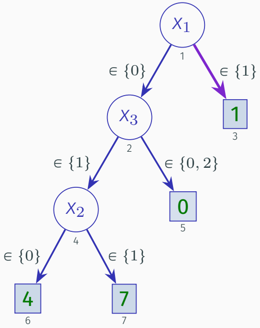

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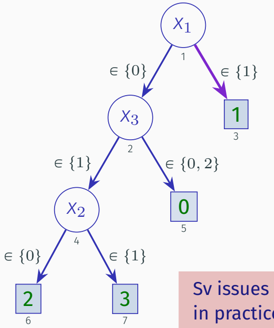

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Sv issues also occur in practice [HM23e]

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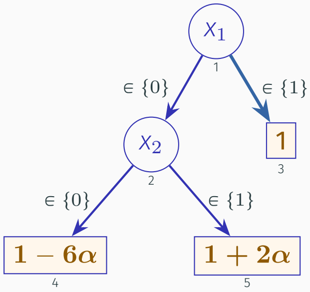
[LHAMS24]



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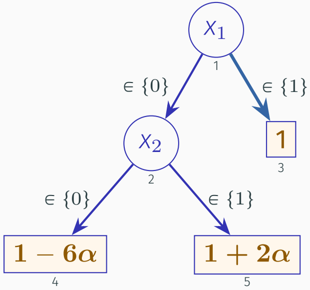

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- **Obs:** $\alpha \neq 1$

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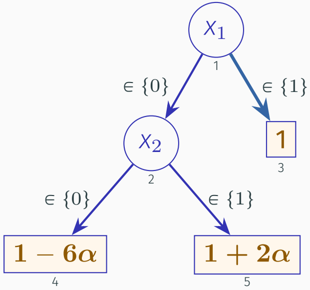

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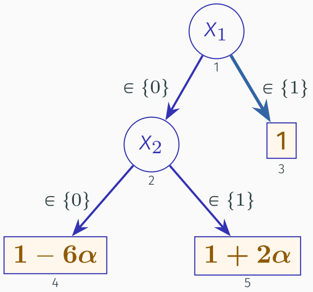

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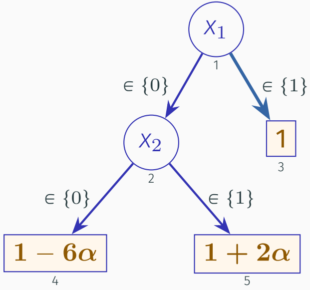

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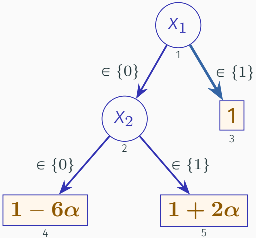
Example devised by O. Letoffe, PhD student at IRIT

More detail

row	x_1	x_2	$\rho(\mathbf{x})$	$\rho_a(\mathbf{x})$ $\alpha = 1/2$	$\rho_b(\mathbf{x})$ $\alpha = 1/4$
1	0	0	$1-6\alpha$	-2	$-1/2$
2	0	1	$1+2\alpha$	2	$3/2$
3	1	0	1	1	1
4	1	1	1	1	1



$\mathcal{S}$	rows($\mathcal{S}$)	$v_e(\mathcal{S})$
$\emptyset$	$1,2,3,4$	$1-\alpha$
$\{x_1\}$	$3,4$	1
$\{x_2\}$	$2,4$	$1+\alpha$
$\{x_1,x_2\}$	4	1

$i=1$					
$\mathcal{S}$	$v_e(\mathcal{S})$	$v_e(\mathcal{S}\cup\{1\})$	$\Delta_1(\mathcal{S})$	$\varsigma(\mathcal{S})$	$\varsigma(\mathcal{S})\times\Delta_1(\mathcal{S})$
$\emptyset$	$1-\alpha$	1	α	$1/2$	$\alpha/2$
$\{2\}$	$1+\alpha$	1	$-\alpha$	$1/2$	$-\alpha/2$
			$\text{Sc}_E(1)$	$=$	0

$i=2$					
$\mathcal{S}$	$v_e(\mathcal{S})$	$v_e(\mathcal{S}\cup\{2\})$	$\Delta_2(\mathcal{S})$	$\varsigma(\mathcal{S})$	$\varsigma(\mathcal{S})\times\Delta_2(\mathcal{S})$
$\emptyset$	$1-\alpha$	$1+\alpha$	2α	$1/2$	α
$\{1\}$	1	1	0	$1/2$	0
			$\text{Sc}_E(2)$	$=$	α

Outline – Unit #07

Corrected SHAP scores & feature importance scores

[LHMS24, LHAMS24]

- Is the theory of Shapley values incorrect?

Corrected SHAP scores & feature importance scores

[LHMS24, LHAMS24]

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  - In XAI: characteristic function uses the expected value
  - This defines the *marginal contribution* in SHAP lingo...

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  - Feature importance scores: [LHAMS24]
    - Generalize recent axiomatic aggregations [BIL+24]
    - Adapt best known power indices
    - Devise new scores for XAI

An initial compromise

[LHAMS24]

- Replace the characteristic function used for SHAP scores:

$$v_e(\mathcal{S}) \quad := \quad \mathbf{E}[\tau(\mathbf{x})\,|\,\mathbf{x}_{\mathcal{S}} = \mathbf{v}_{\mathcal{S}}]$$

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$$\sigma(\mathbf{x}) = \begin{cases} 1 & \text{if } (\kappa(\mathbf{x}) = \kappa(\mathbf{v})) \\ 0 & \text{otherwise} \end{cases}$$

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- Developed SSHAP prototype using SHAP's code base [LHMS24]

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- Corrected SHAP scores reveal tight connection between XAI by feature selection (i.e. WAXps) and feature attribution

Outline – Unit #07

Recap: weighted voting games

- General set up of weighted voting games:
  - Assembly $\mathcal{A}$ of voters, with $m = |\mathcal{A}|$
  - Each voter $i \in \mathcal{A}$ votes Yes with n_i votes; otherwise no votes are counte (and he/she votes No)
  - A coalition is a subset of voters, $\mathcal{C} \subseteq \mathcal{A}$
  - Quota q is the sum of votes required for a proposal to be approved
    - Coalitions leading to sums not less than q are **winning** coalitions
  - A weighted voting game (WVG) is a tuple $[q; n_1, \ldots, n_m]$
    - Example: $[12; 4, 4, 4, 2, 2, 1]$
  - Problem: find a measure of importance of each voter !
    - I.e. measure the a priori voting power of each voter

What are power indices?

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  - Banzhaf [BI65]
  - Coleman [Col71]
  - Johnston [Joh78]
  - Deegan-Packel [DP78]
  - Holler-Packel [HP83]
  - Andjiga [ACL03]
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  - ...
- What characterizes power indices?
  - Account for the cases when voter is *critical* for a winning coalition
    - E.g. in previous example, Luxembourg is never critical for a winning coalition
  - Account for whether coalition is subset-minimal or cardinality-minimal

Towards defining power indices

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- Understanding (subset-)minimal winning coalitions:

Towards defining power indices

- Understanding criticality (used at least since 1954): [SS54]
  - Since the work of Shapley-Shubik [SS54], the criticality of a voter has been accounted for: *"Our definition of the power of an individual member depends on the chance he has of being critical to the success of a winning coalition."*
  - This means that a voter i is critical when:
    - If the voter votes Yes, then we have a winning coalition; and
    - If the voter votes No, then we have a losing coalition.
- Understanding (subset-)minimal winning coalitions:
  - A winning coalition is subset-minimal if removing any single voter results in a losing coalition

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- Understanding (subset-)minimal winning coalitions:
  - A winning coalition is subset-minimal if removing any single voter results in a losing coalition
  - A winning coalition is cardinality-minimal if it has the smallest cardinality among subset-minimal winning coalitions
  - Recall that minimal winning coalitions can be obtained by computing the AXps of a monotonically increasing boolean classifier

Example power indices I

[LHAMS24]

- Necessary definitions (using formal XAI notation...):

$$\mathbb{WA}_i(\mathcal{E}) = \{\mathcal{S} \subseteq \mathcal{F} \,|\, \text{WAXp}(\mathcal{S};\mathcal{E}) \wedge i \in \mathcal{S}\}$$
$$\mathbb{WC}_i(\mathcal{E}) = \{\mathcal{S} \subseteq \mathcal{F} \,|\, \text{WCXp}(\mathcal{S};\mathcal{E}) \wedge i \in \mathcal{S}\}$$
$$\mathbb{A}_i(\mathcal{E}) = \{\mathcal{S} \subseteq \mathcal{F} \,|\, \text{AXp}(\mathcal{S};\mathcal{E}) \wedge i \in \mathcal{S}\}$$
$$\mathbb{C}_i(\mathcal{E}) = \{\mathcal{S} \subseteq \mathcal{F} \,|\, \text{CXp}(\mathcal{S};\mathcal{E}) \wedge i \in \mathcal{S}\}$$

  - Definitions of $\mathbb{WA}$, $\mathbb{WC}$, $\mathbb{A}$, and $\mathbb{C}$ mimic the ones above, but without specifying a voter

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  - Definitions of $\mathbb{WA}$, $\mathbb{WC}$, $\mathbb{A}$, and $\mathbb{C}$ mimic the ones above, but without specifying a voter

- Power indices of Holler-Packel and Deegan-Packel: [HP83, DP78]

$$\mathsf{Sc}_H(i;\mathcal{E}) = \sum\nolimits_{\mathcal{S}\in\mathbb{A}_i(\mathcal{E})} \left({}^{1}/_{|\mathbb{A}(\mathcal{E})|}\right)$$
$$\mathsf{Sc}_D(i;\mathcal{E}) = \sum\nolimits_{\mathcal{S}\in\mathbb{A}_i(\mathcal{E})} \left({}^{1}/_{(|\mathcal{S}| \times |\mathbb{A}(\mathcal{E})|)}\right)$$

Example power indices I

[LHAMS24]

- Necessary definitions (using formal XAI notation...):

$$
\begin{aligned}
\mathbb{WA}_i(\mathcal{E}) &= \{\mathcal{S} \subseteq \mathcal{F} \,|\, \mathsf{WAXp}(\mathcal{S};\mathcal{E}) \land i \in \mathcal{S}\} \\
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\mathbb{C}_i(\mathcal{E}) &= \{\mathcal{S} \subseteq \mathcal{F} \,|\, \mathsf{CXp}(\mathcal{S};\mathcal{E}) \land i \in \mathcal{S}\}
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[HP83, DP78]

$$
\begin{aligned}
\mathsf{Sc}_H(i;\mathcal{E}) &= \sum\nolimits_{\mathcal{S}\in\mathbb{A}_i(\mathcal{E})} \left({}^1/|\mathbb{A}(\mathcal{E})|\right) \\
\mathsf{Sc}_D(i;\mathcal{E}) &= \sum\nolimits_{\mathcal{S}\in\mathbb{A}_i(\mathcal{E})} \left({}^1/(|\mathcal{S}| \times |\mathbb{A}(\mathcal{E})|)\right)
\end{aligned}
$$

  - **Obs:** One *only* needs the AXps

Example power indices II

- Additional definitions:

$$\text{Crit}(i, \mathcal{S}; \mathcal{E}) \quad := \quad \text{WAXp}(\mathcal{S}; \mathcal{E}) \land \neg\text{WAXp}(\mathcal{S} \backslash \{i\}; \mathcal{E})$$

Example power indices II

- Additional definitions:

$$\mathsf{Crit}(i,\mathcal{S};\mathcal{E}) \;:=\; \mathsf{WAXp}(\mathcal{S};\mathcal{E}) \land \neg\mathsf{WAXp}(\mathcal{S}\backslash\{i\};\mathcal{E})$$

- Power indices of Shapley-Shubik, Banzhaf and Johnston: [SS54, BI65, Joh78]

$$\begin{aligned}
\mathsf{Sc}_S(i;\mathcal{E}) &= \sum\nolimits_{\mathcal{S}\subseteq\mathcal{F}\land\mathsf{Crit}(i,\mathcal{S};\mathcal{E})}\left({}^{1}\!/\!\left(|\mathcal{F}|\times\binom{|\mathcal{F}|-1}{|\mathcal{S}|-1}\right)\right)\\
\mathsf{Sc}_B(i;\mathcal{E}) &= \sum\nolimits_{\mathcal{S}\subseteq\mathcal{F}\land\mathsf{Crit}(i,\mathcal{S};\mathcal{E})}\left({}^{1}\!/2^{|\mathcal{F}|-1}\right)\\
\mathsf{Sc}_J(i;\mathcal{E}) &= \sum\nolimits_{\mathcal{S}\subseteq\mathcal{F}\land\mathsf{Crit}(i,\mathcal{S};\mathcal{E})}\left({}^{1}\!/\Delta(\mathcal{S})\right)
\end{aligned}$$

- Additional definitions:

$$\mathsf{Crit}(i,\mathcal{S};\mathcal{E}) \quad := \quad \mathsf{WAXp}(\mathcal{S};\mathcal{E}) \land \neg\mathsf{WAXp}(\mathcal{S}\backslash\{i\};\mathcal{E})$$

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\mathsf{Sc}_S(i;\mathcal{E}) &= \textstyle\sum_{\mathcal{S}\subseteq\mathcal{F}\land\mathsf{Crit}(i,\mathcal{S};\mathcal{E})} \left({}^{1}/\left(|\mathcal{F}|\times\binom{|\mathcal{F}|-1}{|\mathcal{S}|-1}\right)\right)\\
\mathsf{Sc}_B(i;\mathcal{E}) &= \textstyle\sum_{\mathcal{S}\subseteq\mathcal{F}\land\mathsf{Crit}(i,\mathcal{S};\mathcal{E})} \left({}^{1}/{2^{|\mathcal{F}|-1}}\right)\\
\mathsf{Sc}_J(i;\mathcal{E}) &= \textstyle\sum_{\mathcal{S}\subseteq\mathcal{F}\land\mathsf{Crit}(i,\mathcal{S};\mathcal{E})} \left({}^{1}/{\Delta(\mathcal{S})}\right)
\end{aligned}$$

  - One needs the WAXps to find critical voters...

Example #01

- WVG: $[9; 9, 2, 2, 2, 2, 1, 1]$

Example #01

- WVG: $[9; 9, 2, 2, 2, 2, 1, 1]$

- AXps:

$$\begin{array}{lllll} 1 & & & & \\ 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 7 \end{array}$$

Example #01

- WVG: $[9; 9, 2, 2, 2, 2, 1, 1]$
- AXps:

$$\begin{array}{ccccc} 1 & & & & \\ 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 7 \end{array}$$

- Holler-Packel scores: $\langle 0.333, 0.667, 0.667, 0.667, 0.667, 0.333, 0.333 \rangle$
- Banzhaf scores (normalized): $\langle 0.813, 0.040, 0.040, 0.040, 0.040, 0.013, 0.013 \rangle$
- Shapley-Shubik scores: $\langle 0.810, 0.043, 0.043, 0.043, 0.043, 0.010, 0.010 \rangle$
- Different relative orders of voter importance... which ones seem more realistic?

Example #02

- WVG: $[16; 10, 6, 4, 2, 2]$

Example #02

- WVG: $[16; 10, 6, 4, 2, 2]$
- AXps:

$$\begin{array}{ccc} 1 & 2 & \\ 1 & 3 & 4 \\ 1 & 3 & 5 \end{array}$$

Example #02

- WVG: $[16; 10, 6, 4, 2, 2]$
- AXps:

$$\begin{array}{ccc} 1 & 2 & \\ 1 & 3 & 4 \\ 1 & 3 & 5 \end{array}$$

- Deegan-Packel scores: $\langle 0.389, 0.167, 0.222, 0.111, 0.111 \rangle$
- Banzhaf scores (normalized): $\langle 0.524, 0.238, 0.143, 0.048, 0.048 \rangle$
- Shapley-Shubik scores: $\langle 0.617, 0.200, 0.117, 0.033, 0.033 \rangle$
- Different relative orders of voter importance... which ones seem more realistic?

Example #03

- WVG: $[6; 4, 2, 1, 1, 1, 1]$

Example #03

- WVG: $[6; 4, 2, 1, 1, 1, 1]$

- AXps:

$$\begin{array}{lllll} 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 4 \\ 1 & 4 & 5 \\ 1 & 4 & 6 \\ 1 & 3 & 6 \\ 1 & 5 & 6 \\ 1 & 2 \\ 1 & 3 & 5 \end{array}$$

Example #03

- WVG: $[6; 4, 2, 1, 1, 1, 1]$
- AXps:

$$\begin{array}{ccccc} 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 4 \\ 1 & 4 & 5 \\ 1 & 4 & 6 \\ 1 & 3 & 6 \\ 1 & 5 & 6 \\ 1 & 2 \\ 1 & 3 & 5 \end{array}$$

- Deegan-Packel scores: $\langle 0.312, 0.087, 0.150, 0.150, 0.150, 0.150 \rangle$
- Banzhaf scores (normalized): $\langle 0.542, 0.125, 0.083, 0.083, 0.083, 0.083 \rangle$
- Shapley-Shubik scores: $\langle 0.533, 0.133, 0.083, 0.083, 0.083, 0.083 \rangle$
- Different relative orders of voter importance... which ones seem more realistic?

Example #04

- WVG: $[21; 12, 9, 4, 4, 1, 1, 1]$

Example #04

- WVG: $[21; 12, 9, 4, 4, 1, 1, 1]$
- AXps:

$$\begin{array}{cccc} 1 & 2 & & \\ 1 & 3 & 4 & 5 \\ 1 & 3 & 4 & 6 \\ 1 & 3 & 4 & 7 \end{array}$$

Example #04

- WVG: $[21; 12, 9, 4, 4, 1, 1, 1]$
- AXps:

$$\begin{array}{cccc} 1 & 2 & & \\ 1 & 3 & 4 & 5 \\ 1 & 3 & 4 & 6 \\ 1 & 3 & 4 & 7 \end{array}$$

- Deegan-Packel scores: $\langle 0.312, 0.125, 0.188, 0.188, 0.062, 0.062, 0.062 \rangle$
- Banzhaf scores (normalized): $\langle 0.481, 0.309, 0.086, 0.086, 0.012, 0.012, 0.012 \rangle$
- Shapley-Shubik scores: $\langle 0.574, 0.257, 0.074, 0.074, 0.007, 0.007, 0.007 \rangle$
- Different relative orders of voter importance... which ones seem more realistic?

Outline – Unit #07

Exact Shapley Values for XAI

Myth #03: Shapley Values for XAI

Corrected SHAP Scores

Voting Power & Power Indices

Feature Importance Scores

From power indices to feature importance scores

- A **Feature Importance Score** (FIS) is a measure of feature importance in XAI, parameterizable on an **explanation problem** and a chosen **characteristic function**
  - Explanation problem: $(\mathcal{M}, (\mathbf{v}, q))$
  - Define characteristic function using explanation problem (more next slide)

- Obs: Can adapt (generalized) power indices as templates for feature importance scores

- Obs: Can devise new templates and/or new FISs

Some examples (1 of 2)

- More notation:

$$\Delta_i(\mathcal{S};\mathcal{E},\upsilon) = \upsilon(\mathcal{S};\mathcal{E}) - \upsilon(\mathcal{S}\setminus\{i\};\mathcal{E})$$

  - Can use **any** characteristic function, including those presented earlier in this lecture

Some examples (1 of 2)

- More notation:

$$\Delta_i(\mathcal{S};\mathcal{E},\upsilon) = \upsilon(\mathcal{S};\mathcal{E}) - \upsilon(\mathcal{S}\setminus\{i\};\mathcal{E})$$

  - Can use **any** characteristic function, including those presented earlier in this lecture

- Some templates:
  - Shapley-Shubik:

$$\mathsf{TSc}_S(i;\mathcal{E},\upsilon) := \sum_{\mathcal{S}\in\{\mathcal{T}\subseteq\mathcal{F}\,|\,i\in\mathcal{T}\}} \left(\frac{\Delta_i(\mathcal{S};\mathcal{E},\upsilon)}{|\mathcal{F}|\times\binom{|\mathcal{F}|-1}{|\mathcal{S}|-1}}\right)$$

  - Banzhaf:

$$\mathsf{TSc}_B(i;\mathcal{E},\upsilon) := \sum_{\mathcal{S}\in\{\mathcal{T}\subseteq\mathcal{F}\,|\,i\in\mathcal{T}\}} \left(\frac{\Delta_i(\mathcal{S};\mathcal{E},\upsilon)}{2^{|\mathcal{F}|-1}}\right)$$

Some examples (1 of 2)

- More notation:

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  - Banzhaf:

$$\mathsf{TSc}_B(i;\mathcal{E},\upsilon) := \sum_{\mathcal{S}\in\{\mathcal{T}\subseteq\mathcal{F}\,|\,i\in\mathcal{T}\}} \left(\frac{\Delta_i(\mathcal{S};\mathcal{E},\upsilon)}{2^{|\mathcal{F}|-1}}\right)$$

- Can use other templates

Some examples (1 of 2)

- More notation:

$$\Delta_i(\mathcal{S};\mathcal{E},\upsilon) = \upsilon(\mathcal{S};\mathcal{E}) - \upsilon(\mathcal{S}\setminus\{i\};\mathcal{E})$$

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- Some templates:
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  - Banzhaf:

$$\text{TSc}_B(i;\mathcal{E},\upsilon) := \sum_{\mathcal{S}\in\{\mathcal{T}\subseteq\mathcal{F}\,|\,i\in\mathcal{T}\}} \left(\frac{\Delta_i(\mathcal{S};\mathcal{E},\upsilon)}{2^{|\mathcal{F}|-1}} \right)$$

- Can use other templates

- Can devise FISs without exploiting existing templates

Some examples (2 of 2)

- Recall WAXp based characteristic function:

$$v_a(\mathcal{S}) \quad := \quad \begin{cases} 1 & \text{if } \mathbf{E}[\sigma(\mathbf{x}) \,|\, \mathbf{x}_{\mathcal{S}} = \mathbf{v}_{\mathcal{S}}] = 1 \\ 0 & \text{otherwise} \end{cases}$$

Some examples (2 of 2)

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- Some FISs:
  - Shapley-Shubik:

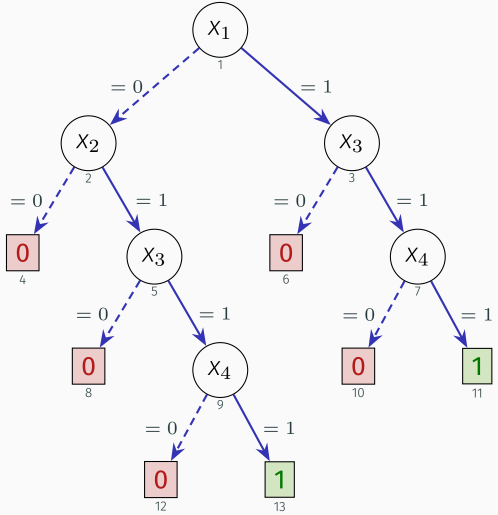
$$\mathsf{Sc}_S(i;\mathcal{E}) := \mathsf{TSc}_S(i;\mathcal{E},v_a) := \sum_{\mathcal{S}\in\{\mathcal{T}\subseteq\mathcal{F}\,|\,i\in\mathcal{T}\}} \left(\frac{\Delta_i(\mathcal{S};\mathcal{E},v_a)}{|\mathcal{F}| \times \binom{|\mathcal{F}|-1}{|\mathcal{S}|-1}} \right)$$

  - Banzhaf:

$$\mathsf{Sc}_B(i;\mathcal{E}) := \mathsf{TSc}_B(i;\mathcal{E},v_a) := \sum_{\mathcal{S}\in\{\mathcal{T}\subseteq\mathcal{F}\,|\,i\in\mathcal{T}\}} \left(\frac{\Delta_i(\mathcal{S};\mathcal{E},v_a)}{2^{|\mathcal{F}|-1}} \right)$$

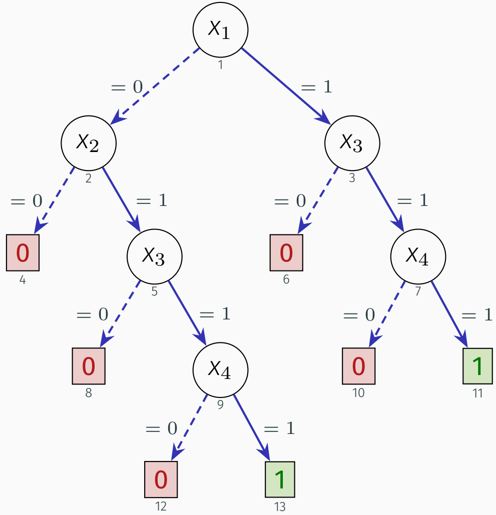
A concrete example

- AXps: $\{\{1,3,4\},\{2,3,4\}\}$
- Feature attribution:



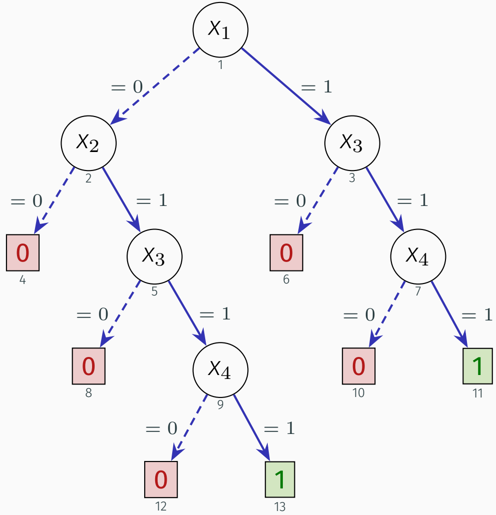
A concrete example

- AXps: $\{\{1,3,4\},\{2,3,4\}\}$
- Feature attribution:
  - SS: $\langle 0.083, 0.083, 0.417, 0.417 \rangle$



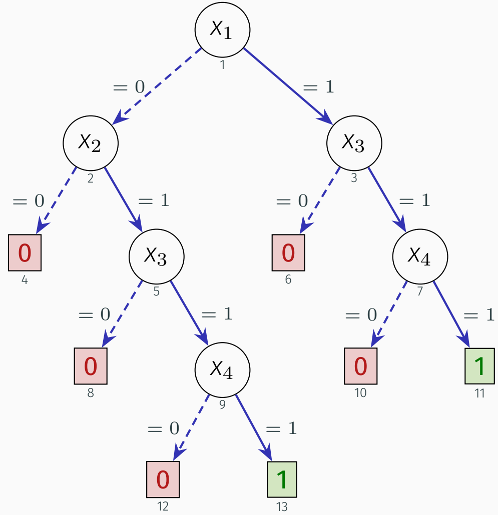
A concrete example

- AXps: $\{\{1,3,4\},\{2,3,4\}\}$
- Feature attribution:
  - SS: $\langle 0.083, 0.083, 0.417, 0.417\rangle$
  - B (norm.): $\langle 0.125, 0.125, 0.375, 0.375\rangle$



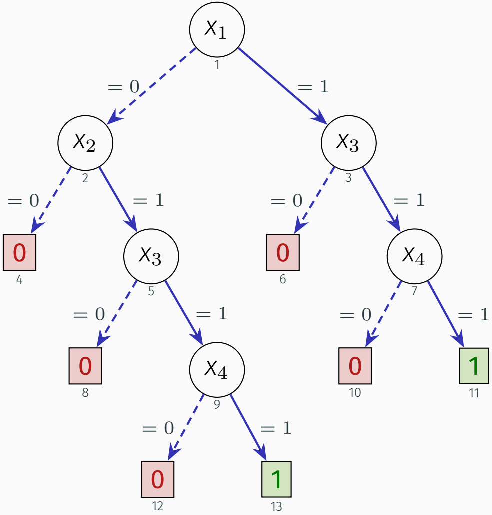
A concrete example

- AXps: $\{\{1, 3, 4\}, \{2, 3, 4\}\}$
- Feature attribution:
  - SS: $\langle 0.083, 0.083, 0.417, 0.417 \rangle$
  - B (norm.): $\langle 0.125, 0.125, 0.375, 0.375 \rangle$
  - J (norm.): $\langle 0.111, 0.111, 0.389, 0.389 \rangle$



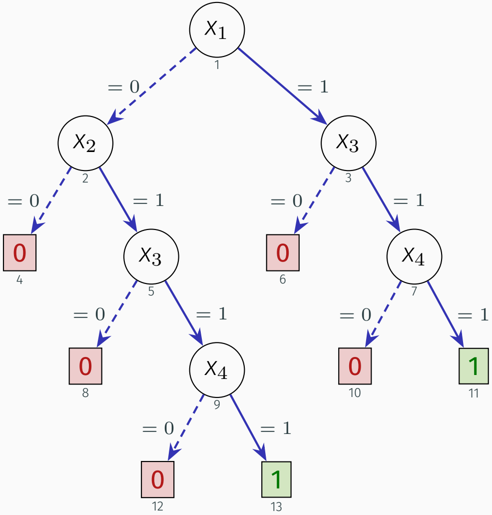
A concrete example

- AXps: $\{\{1,3,4\},\{2,3,4\}\}$
- Feature attribution:
  - SS: $\langle 0.083, 0.083, 0.417, 0.417\rangle$
  - B (norm.): $\langle 0.125, 0.125, 0.375, 0.375\rangle$
  - J (norm.): $\langle 0.111, 0.111, 0.389, 0.389\rangle$
  - HP: $\langle 0.167, 0.167, 0.333, 0.333\rangle$



A concrete example

- AXps: $\{\{1,3,4\},\{2,3,4\}\}$
- Feature attribution:
  - SS: $\langle 0.083, 0.083, 0.417, 0.417\rangle$
  - B (norm.): $\langle 0.125, 0.125, 0.375, 0.375\rangle$
  - J (norm.): $\langle 0.111, 0.111, 0.389, 0.389\rangle$
  - HP: $\langle 0.167, 0.167, 0.333, 0.333\rangle$
  - DP: $\langle 0.167, 0.167, 0.333, 0.333\rangle$



Questions?

Unit #08

Conclusions & Research Directions

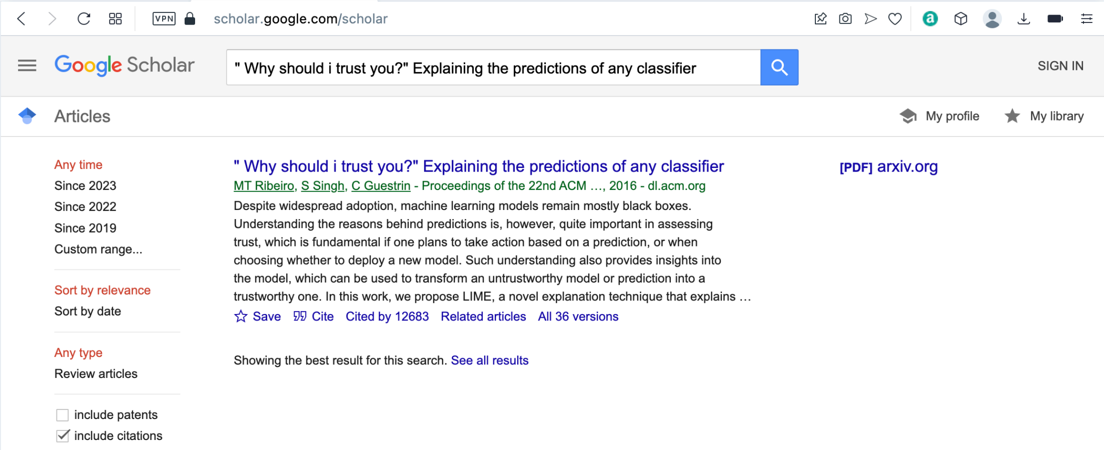
Outline – Unit #08

Some Words of Concern

Conclusions & Research Directions

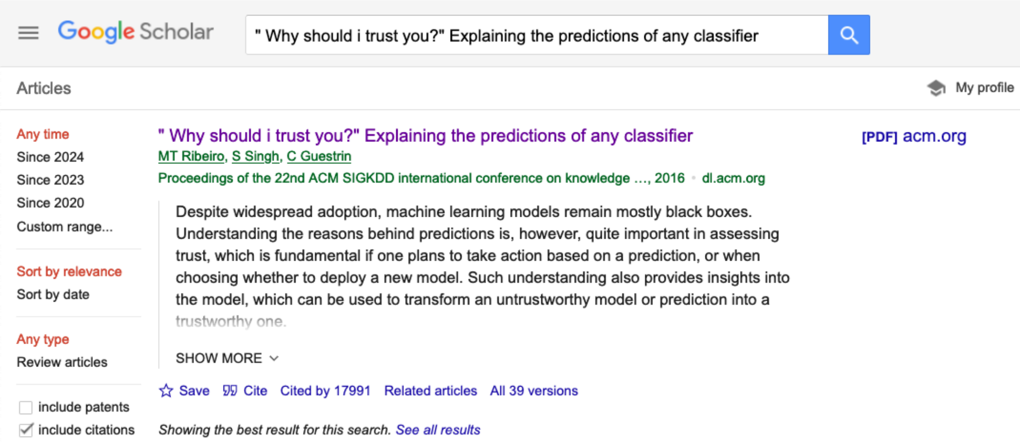
Can heuristic XAI's myths be stopped?

LIME on 2023/05/31:



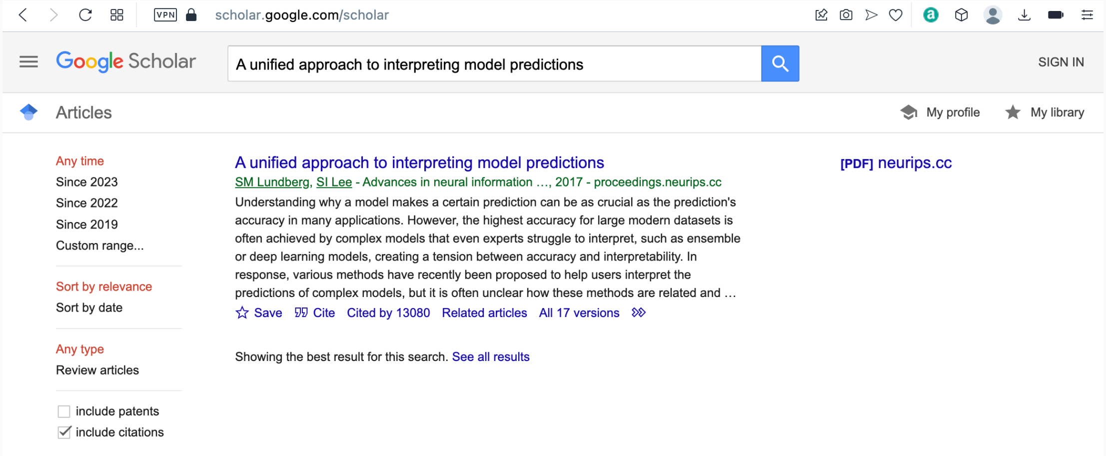
Can heuristic XAI's myths be stopped?

LIME on 2024/07/02:

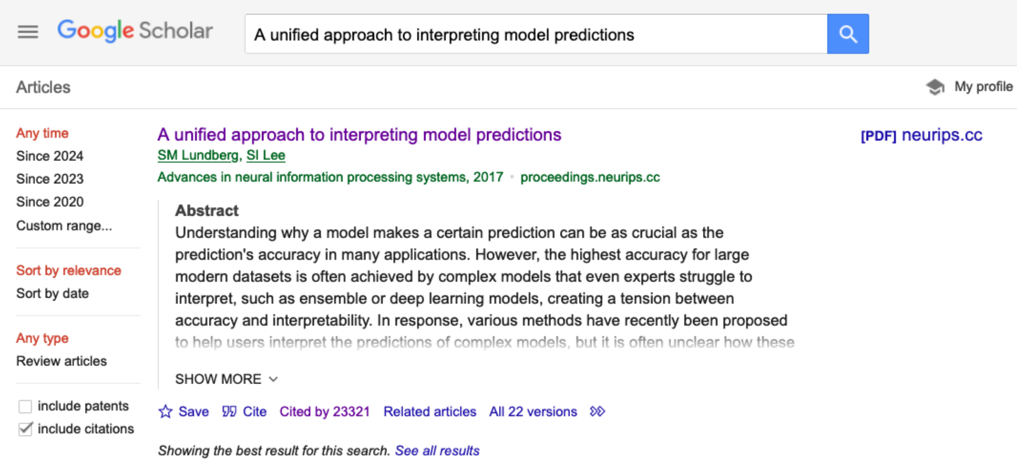


Can heuristic XAI’s myths be stopped?

SHAP on 2023/05/31:



SHAP on 2024/07/02:



What's the bottom line?

- (Heuristic) XAI research experiences a persistent "*Don't Look Up*" moment...

- (Heuristic) XAI research experiences a persistent "*Don't Look Up*" moment...

BTW, there are a multitude of proposed uses of LIME/SHAP in medicine... ⚠

Some unsettling works...

- For DTs:
  - One AXp in polynomial-time [IIM20, HIIM21, IIM22]
  - All CXps in polynomial-time [HIIM21, IIM22]

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Declarative Reasoning on Explanations Using Constraint Logic Programming

Abstract. Explaining opaque Machine Learning (ML) models is an increasingly relevant problem. Current explanation in AI (XAI) methods suffer several shortcomings, among others an insufficient incorporation of background knowledge, and a lack of abstraction and interactivity with the user. We propose REASONX, an explanation method based on Constraint Logic Programming (CLP). REASONX can provide declarative, interactive explanations for decision trees, which can be the ML models under analysis or global/local surrogate models of any black-box model. Users can express background or common sense knowledge using linear constraints and MILP optimization over features of factual and contrastive instances, and interact with the answer constraints at different levels of abstraction through constraint projection. We present here the architecture of REASONX, which consists of a Python layer, closer to the user, and a CLP layer. REASONX's core execution engine is a Prolog meta-program with declarative semantics in terms of logic theories.

arXiv:2309.00422v1 [cs.AI] 1 Sep 2023

- For DTs:
  - One AXp in polynomial-time [IIM20, HIIM21, IIM22]
  - All CXps in polynomial-time [HIIM21, IIM22]

HHAI 2024: Hybrid Human AI Systems for the Social Good
F. Lorig et al. (Eds.)
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doi:10.3233/FAIA240183

Exploring Large Language Models Capabilities to Explain Decision Trees

Some unsettling works...

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Explainable Artificial Intelligence for Academic Performance Prediction. An Experimental Study on the Impact of Accuracy and Simplicity of Decision Trees on Causability and Fairness Perceptions

FAccT '24, June 03–06, 2024, Rio de Janeiro, Brazil

ACM ISBN 979-8-4007-0450-5/24/06
https://doi.org/10.1145/3630106.3658953

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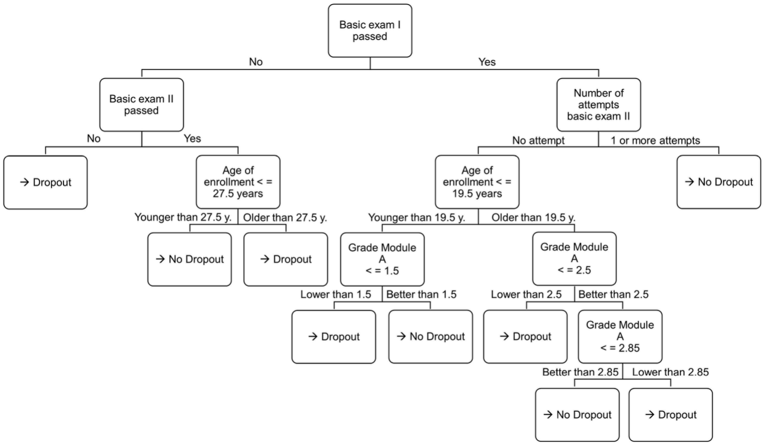

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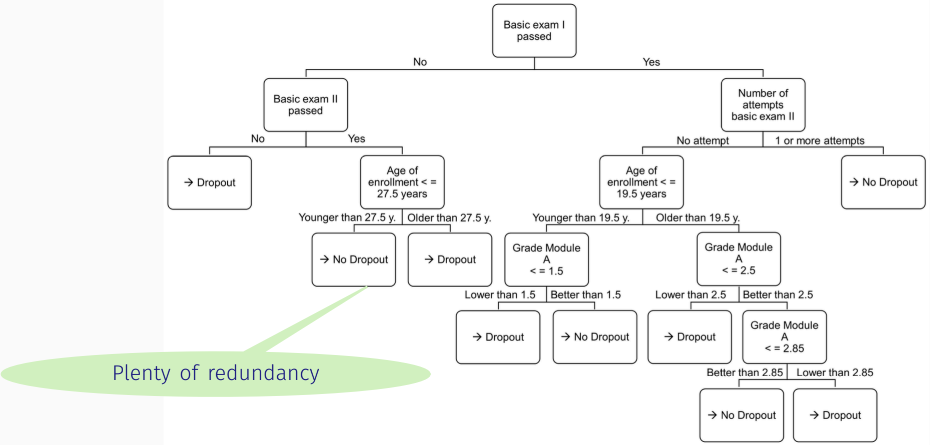
  - One AXp in polynomial-time [IIM20, HIIM21, IIM22]
  - All CXps in polynomial-time [HIIM21, IIM22]



Outline – Unit #08

Some Words of Concern

Conclusions & Research Directions

Conclusions

- Covered logic-based (aka symbolic, aka formal) XAI & its recent progress:
  - Abductive & contrastive explanations
  - Reviewed their computation in practice
  - Duality & enumeration
  - Other explainability queries – feature necessity & relevancy

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- Demonstrated tight connection between (rigorous) feature selection and (rigorous) feature attribution in XAI
- Symbolic XAI exhibits links with many fields of research: machine learning, artificial intelligence, formal methods, automated reasoning, optimization, computational social choice (& game theory), etc.

Research directions

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- Scalabilitty, scalability, and scalability

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- Scalabilitty, scalability, and scalability
- Probabilitistic explanations

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- New topics from discussions with participants of ESSAI'24 – **Thank you!**

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- Scalabilitty, scalability, and scalability
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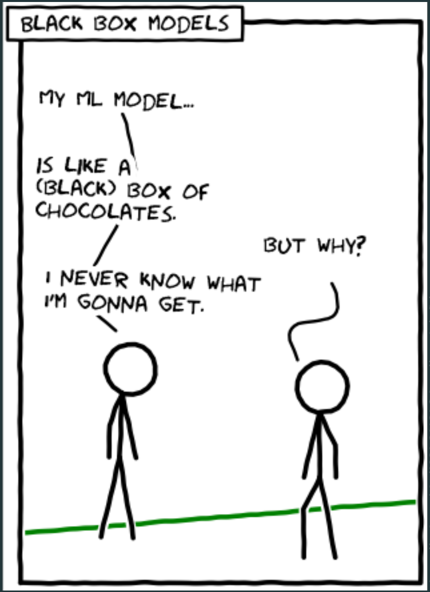
- Rigorous feature attribution
- Preferred explanations
- Certified XAI tools
- New topics from discussions with participants of ESSAI'24 – **Thank you!**
- … And trying to curb the massive momentum of (heuristic) XAI **myths**!

What this course covered

- Lecture 01 – units:
  - #01: Foundations
- Lecture 02 – units:
  - #02: Principles of symbolic XAI – feature selection
  - #03: Tractability in symbolic XAI (& myth of interpretability)
- Lecture 03 – units:
  - #04: Intractability in symbolic XAI (& myth of model-agnostic XAI)
  - #05: Explainability queries
- Lecture 04 – units:
  - #06: Advanced topics
- Lecture 05 – units:
  - #07: Principles of symbolic XAI – feature attribution (& myth of Shapley values in XAI)
  - #08: Conclusions & research directions

Q & A

Acknowledgment: joint work with X. Huang, Y. Izza, O. Létoffé, A. Ignatiev, N. Narodytska, M. Cooper, N. Asher, A. Morgado, J. Planes, et al.



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